JAGIELLONIAN UNIVERSITY

DOCTORAL THESIS

Dynamics of eigenvalues and eigenvectors in non-Hermitian matrix models

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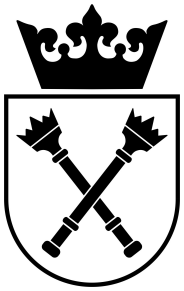
A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy

in the

Theory of Complex Systems Group
Faculty of Physics, Astronomy and Applied Computer Science



June 20, 2016

Wydział Fizyki, Astronomii i Informatyki Stosowanej
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Oświadczenie

Ja niżej podpisany *Jacek Grela* (nr indeksu: 1070128) doktorant Wydziału Fizyki, Astronomii i Informatyki Stosowanej Uniwersytetu Jagiellońskiego oświadczam, że przedłożona przeze mnie rozprawa doktorska pt. *"Dynamics of eigenvalues and eigenvectors in non-Hermitian matrix models"* jest oryginalna i przedstawia wyniki badań wykonanych przeze mnie osobiście, pod kierunkiem prof. *Macieja A. Nowaka*. Pracę napisałem samodzielnie.

Oświadczam, że moja rozprawa doktorska została opracowana zgodnie z Ustawą o prawie autorskim i prawach pokrewnych z dnia 4 lutego 1994 r. (Dziennik Ustaw 1994 nr 24 poz. 83 wraz z późniejszymi zmianami).

Jestem świadom, że niezgodność niniejszego oświadczenia z prawdą ujawniona w dowolnym czasie, niezależnie od skutków prawnych wynikających z ww. ustawy, może spowodować unieważnienie stopnia nabytego na podstawie tej rozprawy.

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Abstract

Dynamics of eigenvalues and eigenvectors in non-Hermitian matrix models

by Jacek GRELA

This dissertation is a study of Hermitian and non-Hermitian dynamical matrices with an emphasis put on the behaviour of their eigenvalues and eigenvectors. The thesis is divided into three main parts — first two chapters are devoted to the Hermitian and non-Hermitian models and the third describes a computational tool used throughout the thesis. We work in a scheme of diffusive dynamics considered by Dyson under which a hydrodynamical picture is realized by the presence of Burgers' equation.

Firstly, we inspect the hydrodynamics of dynamical Hermitian matrices arising in the large matrix size limit — the eigenvalues and eigenvectors evolve almost independently of each other and the former are more important for the dynamics. By considering several Hermitian models, we find a recurring hydrodynamical Burgers' equation for the respective Green's function encoding the eigenvalue information. Implicit solutions of these equations are obtained by the method of complex characteristics. We study the averaged characteristic polynomial and the averaged inverse characteristic polynomial in the dynamical Gaussian Unitary Ensemble, we show how these two objects fix the correlation kernel of the model and we find an universal Airy and Pearcey behaviour in both of these observables. By the use of the collective variables, we re-establish the hydrodynamical picture and we compute the asymptotic forms of Harish-Chandra/Itzykson–Zuber and Berezin–Karpelevich integrals.

Secondly, we turn to the description of dynamical non-Hermitian matrices. We conduct numerical studies to show that the eigenvalues and eigenvectors of these matrices co-evolve and cannot be decoupled. Based on this observation, we recover the hydrodynamical picture in the non-Hermitian class although in a degrees of freedom combining the eigenvalue and eigenvector information. Identifying and interpreting these new variables is crucial in understanding the non-Hermitian random matrices. To this end we derive the non-Hermitian Burgers' equation for the q-Green's function, we find macroscopic formulas for the spectral densities and the eigenvector correlators and we identify the relevance of non-normal initial conditions. Additionally, besides the results found in the large matrix size limit, we compute the formula for the spectral density valid for any matrix size.

Thirdly, we describe an approach which was formalized under the name of the diffusion method. It is a computational tool of finding the averages of determinants. We describe it as a two-step procedure which assumes diffusive dynamics of matrices and look for similar diffusive equations for the observables but now in a dual space of auxiliary variables.

Acknowledgements

I would like to thank my advisor for the support during the preparation and the writing of this thesis, my collaborators for many long, sometimes passionate and always fruitful conversations and my family for helping me throughout my life. All this would not be possible without the support of my girlfriend and my friends.

Contents

Publications

The dissertation is based on the following publications:

1. *Dysonian dynamics of the Ginibre ensemble*, Z. Burda, **J. Grela**, M. A. Nowak, W. Tarnowski, P. Warchoł, Phys. Rev. Lett. **113**, 104102 (2014).
   A brief discussion with main results is given in Chapter 3.2 and the reference is found in (Burda et al., 2014).

2. *Unveiling the significance of eigenvectors in diffusing non-hermitian matrices by identifying the underlying Burgers dynamics*, Z. Burda, **J. Grela**, M. A. Nowak, W. Tarnowski, P. Warchoł, Nucl. Phys. B **897**, 421 (2015).
   Discussion is given in Chapters 3.1, 3.3 and the reference is found in (Burda et al., 2015).

3. *Diffusion in the space of complex Hermitian matrices*, J.-P. Blaizot, **J. Grela**, M. A. Nowak, P. Warchoł, Acta Phys. Pol. B **46**(9), 73 (2015).
   Discussion is given in Chapter 2.3 and the reference is found in (Blaizot et al., 2015a).

4. *Diffusion method in Random Matrix Theory*, **J. Grela**, J. Phys. A: Math. Theor. **49** 015201 (2015).
   Discussion is given in Chapter 4 and the reference is found in (Grela, 2016).

5. *Hydrodynamical spectral evolution for random matrices*, P.J. Forrester, **J. Grela**, J. Phys. A: Math. Theor. **49** 085203 (2016).
   Discussion is given in Chapter 2.5.2 and the reference is found in (Forrester and Grela, 2016).

6. *Ornstein-Uhlenbeck diffusion of hermitian and non-hermitian matrices - unexpected links*, J.-P. Blaizot, J. Grela, M. A. Nowak, W. Tarnowski, P. Warchoł, J. Stat. Mech. Theor. Exp. 5 054037 (2016).
   Discussion is given in Chapters 2.4, 3.4 and the reference is found in (Blaizot et al., 2016).

7. *Exact spectral densities of complex noise-plus-structure random matrices*, **J. Grela**, T. Guhr, to appear in Phys. Rev. E (2016).
   Discussion is given in Chapter 3.5 and the reference is found in (Grela and Guhr, 2016).

Additionally, a list of publications prepared by the author during his PhD studies and not contained in the dissertation:

1. *Universal spectral shock in random matrix theory - lessons for QCD*, J.-P. Blaizot, **J. Grela**, M. A. Nowak, P. Warchoł, Acta Phys. Pol. B **46**(9), 35 (2015).
   The reference is found in (Blaizot et al., 2015b).

Chapter 1

Introduction

1.1 Why random matrices?

When confronted with random matrices for the first time, the concept seems to be both easy to grasp and of limited, if any, real-life application. Admittedly, it is in fact a simple and hardly revolutionary idea — arrange random numbers into a table and study their properties. Formulated as such, it is a merely sub-field of the multivariate probability theory studied thoroughly by mathematicians. Despite these reservations, an interdisciplinary research field arose as an interplay between physics, mathematics and statistics. In an unique way, the matrix became an unifying concept which proved to be useful in applications, served as a last resort tool in hopelessly complex physical systems and have shown inner mathematical beauty. To strengthen this notion, we describe three pioneering works on random matrices which established a tripod on which random matrix theory stands firmly today.

One of the first modern studies on random matrices is due to John Wishart (Wishart, 1928), a statistician who was interested in estimating covariance matrices of real-life data. By considering N random variables drawn from M-dimensional normal distributions he looked at a rectangular data matrix $X_{N\times M}$ and at the joint probability density function of its empirical covariance matrix XX^T. That way he delivered the Wishart distribution nowadays used in Bayesian inference and in estimating covariance matrices. Perhaps more importantly from our point of view, his work laid the foundations of large-scale data analysis used nowadays in finances, telecommunications and big-data science.

Secondly, a physicist Eugene Wigner (Wigner, 1955) wrote a seminal paper on the behaviour of complex quantum systems. His ambition was to explain experimental data of resonant energy spectra of compound nuclei. The route he took was however unorthodox — instead of constructing plausible physical models, a novel "statistical" approach was considered. Because the system at hand is hopelessly complicated, he studied a model Hamiltonian as filled with numbers ± 1 chosen randomly with imposing only proper symmetry conditions (Hermitian, symmetric or other). This can be done since the Hamiltonian can be represented as an infinite matrix. This simple argument produced a surmise for the probability function of the difference s between two consecutive energies:

$$p(s)ds \sim se^{-s^2}ds, \tag{1.1}$$

which was spectacularly confirmed by experimental data and was later identified as the Gaussian Orthogonal Ensemble.

Last but not least, the seminal study of Freeman J. Dyson (Dyson, 1962b; Dyson, 1962c; Dyson, 1962d) put a cornerstone on the mathematical side of random matrices. In a series of papers, three classes of random matrix models organized by their

global symmetries were introduced. The Gaussian Unitary, Orthogonal and Symplectic Ensembles were introduced (the famous Dyson's threefold way) where the matrices are Hermitian, symmetric and self-dual, respectively. He was also considering unitary matrices called Circular Ensembles and thus made a big step in mapping the landscape of random matrix models. In a subsequent paper (Dyson, 1962a) he also introduced dynamical matrix models, an approach which is the main topic of this dissertation.

Aforementioned examples were chosen as three basic constituents of the field — statistics, physics and mathematics and to accentuate the interdisciplinary character of Random Matrix Theory present from its inception(s). Consequently, new instances of random matrices occurring in Nature are found quite frequently. One can therefore ask an important question — why is that the case? It is really baffling since these simple objects seem to gather topics as broad as quantum mechanics (Beenakker, 1997), sketchy bus systems (Krbálek and Šeba, 2000) and prime numbers (Keating and Snaith, 2000). To try and answer this question we shall now touch on the phenomenon of universality.

The property of universality in Random Matrix Theory means that certain classes of observables (on microscopic scales and in the large matrix size limit) have universal forms dependent only on the symmetries (and not on the underlying probability distribution). Although it is formally a conjecture and a lot of work is devoted to find its boundaries (f.e. by studying the class of Wigner matrices), it was proved in many simple cases and many physicists claim it is now an established fact. To answer the question of abundance, we comment on a different phenomenon with the same name.

Universality in statistical physics near critical points is a property of the system where only a set of fixed numbers called critical exponents gives a full description without any regard to microscopic interactions. These numbers depend only on general properties like dimensionality and the symmetries and thus bear much resemblance to the namesake of random matrices. As a consequence, a priori two completely different systems — two-dimensional Ising model and the (3,4) Conformal Field Theory are equivalent (near the criticality) since they share the same set of critical exponents (i.e. form the same universality class). An analogous case can be made by studying the statistics of (non-trivial) Riemann zeroes and eigenvalues of Gaussian Unitary Ensemble — their descriptions coincide because they lie in the same random matrix universality class. From this point of view, the observed abundance of random matrix applications lies in the fact that these models turn out as the simplest representatives of a handful of universality classes and thus random matrix models effectively map the landscape of complex phenomena of astonishing variety.

1.2 Why so dynamical?

If random matrices are worth studying, why do we need to introduce the dynamics? As we argued before, randomness is the general least effort approach and introducing it again seems to be the wrong direction, just after we argued how hopelessly complex problems are tractable when we ignore the gruesome details. This approach has however a more theoretical flavour to it — since we know that the eigenvalues of random matrices are drawn from a certain joint probability density function, is there a microscopic model such that it reproduces this density? The question is similar to the one which Boltzmann asked when statistical mechanics was conceived. The known laws of thermodynamics are describing only the macroscopic phenomena. What is however the nature of the underlying microscopic world from which the macroscopic laws

follow? The question Dyson posed in (Dyson, 1962a) is perhaps not as bold although it shares a similar spirit - he was interested in the underlying dynamics of random matrix eigenvalues which would produce the static models he calculated earlier. Despite his efforts in finding Newtonian laws (as Boltzmann who had an atomistic picture in mind), he found instead a stochastic type dynamics (i.e. non-Newtonian) reproducing the correct joint probability density functions.

The time parameter in which discussed dynamics takes place is often called fictitious to emphasize its status. In applications it can be identified with an area of the Wilson loop (QCD), the length of a conductor (quantum transport) or a magnetic field (particles in strong magnetic fields). As we delve into purely theoretical considerations, we did not address the possible interpretations of the dynamics.

Because the procedure of Dyson seems trivial, we address the confusion which may arise when we infer the dynamical behaviour by probing its static features and thus comment on possible pitfalls of identifying dynamical and static random matrix models. To this end, consider a harmonic oscillator as a dynamical toy-model with position fulfilling the equation $x(t) = \sin t$. Although the model is one-dimensional and lacks any stochastic features, it will suffice to make the point. We know well that the particle is confined in a quadratic confining potential with a minimum value at zero. Suppose however that we ignore the context and look at the trajectory $x(t)$ alone. Then, we can deduce qualitative properties of the potential — the particle moves in a confining well since the motion is periodic and from the *average* position $\langle x \rangle = 0$ we read off the minimum of the potential. We can however approach it differently and look at the *most probable* position $\langle \delta(x(t) - u) \rangle$ which in our case reads $\frac{1}{\pi}(1 - u^2)^{-1/2}$. Now the $u = 0$ position is the least probable and so a *repulsive* potential near the origin should be present to account for that. This erroneous conclusion is made since we used the static information to infer the underlying dynamics. Resolution is simple if the velocity is taken into account - the particle is found near zero not as often as at the ± 1 edges exactly because exactly at that point it has the largest velocity. The basic conclusions of this simple example hold also in the more involved cases when $x(t)$ is multi-dimensional and stochastic.

1.3 Random matrix theory crash course

We discuss shortly the basic facts of random matrices and define some of its most fundamental objects with special emphasis on their usage in this dissertation.

Spectral density. As the matrices comprise of their eigenvalues and eigenvectors, likewise a random matrix is characterized by a distribution of its eigenvalues and eigenvectors. To probe the probability density function of the eigenvalues we define a spectral density (or one-point correlation function) $\rho_N(z)$:

$$\rho_N(z) = \frac{1}{N} \left\langle \sum_{i=1}^{N} \delta(z - z_i) \right\rangle, \tag{1.2}$$

where the averaging is done over the random matrix of choice and z_i's are its N eigenvalues. Dirac delta functions enumerate the eigenvalues and the object is normalized $\int \rho_N(z) = 1$. The eigenvalues z_i can be either a general complex, real, positive numbers or a pure phase, depending on the matrix.

Green's function. Often the spectral density (1.2) is not as useful as the Green's function (also known as the resolvent or the Stieltjes transform):

$$G_N(z) = \frac{1}{N}\left\langle \sum_{i=1}^{N} \frac{1}{z - z_i} \right\rangle, \tag{1.3}$$

with the same notation as before. These two observables contain the same information on the positions of eigenvalues — before taking the average, the spectral density comprise of delta-like peaks always when $z = z_i$ and the Green's function has a simple pole at the same points. They are related by the Sochocki-Plemejl formula

$$\rho_N(z) = \frac{1}{2\pi i} \lim_{\epsilon \to 0^+} \left(G_N(z - i\epsilon) - G_N(z + i\epsilon) \right). \tag{1.4}$$

Large matrix size limit. Most of the results in Random Matrix Theory is obtained in the regime where the matrix size goes to infinity (called also the large N limit). This limit has originated in the works of t'Hooft (Hooft, 1974) who looked at the $U(N)$ gauge theories in the limit where the group space $N \to \infty$ and identified with it a topological expansion. A mundane reason of taking it is that predominantly the large N formulae simplify considerably. Lastly, the questions of universality are answered only when matrix size is not a parameter. In this dissertation lots of formulas will be given in this limit and often the lack of "N" subscript in the formula means that this limit was taken beforehand.

Characteristic polynomial. Last object is the characteristic polynomial (or characteristic determinant) defined as

$$U_N(z) = \left\langle \prod_{i=1}^{N} (z - z_i) \right\rangle, \tag{1.5}$$

where the notation is as before. Although its interpretation is not as clear as the spectral density (1.2) and the Green's function (1.3), the information about the positions of eigenvalues is likewise contained in it. In most matrix models, in the large N limit it reduces to the Green's function via a transform

$$G(z) = \lim_{N \to \infty} \frac{1}{N} \partial_z \ln U_N(z), \tag{1.6}$$

where $G(z) \equiv \lim_{N \to \infty} G_N(z)$.

1.4 Summary of the thesis

Thesis consists of two main sections dealing with two different yet relatable worlds: of dynamical random matrices and dynamical non-Hermitian random matrices. The first part describes several models whose eigenvalues are confined to one spatial dimension. Whether it is a real line, a half-line or a unit circle, the dynamical approach is successful in describing them. The second part discusses non-Hermitian dynamical models which are two-dimensional instead. Unexpectedly, the dynamical framework is also indispensable in the description of these matrices. In what follows we report on the main results and sketch the motivation behind this work.

Dynamical random matrices. Initial impetus to this line of inquiry is traced back to the seminal works of Dyson (Dyson, 1962a) who first became aware of the type of dynamics one needs to impose on matrices in order to draw a link with Random Matrix Theory. The correct approach was a stochastic motion where the entries of the matrix are the random variables. In that spirit, we start off in Chapter 2.1 by sketching this dynamical framework in one dimension to arrive at the celebrated Smoluchowski–Fokker–Planck equation and other equivalent formulations. After generalizing this setting to dynamical matrices, we discuss on how the independent stochastic dynamics of matrices induce a motion in the space of eigenvalues and eigenvectors in the dynamical Gaussian β Ensembles. The straightforward conclusion is that the eigenvalues of the dynamical Gaussian Unitary Ensemble repel each other by a $1/x$ interaction term and their evolution is independent of the eigenvectors. In turn, the corresponding eigenvectors do depend on the eigenvalues but the relation is only one way.

Another source of inspiration was drawn from several recent papers of Nowak and collaborators (Gudowska-Nowak et al., 2003; Blaizot and Nowak, 2008; Blaizot and Nowak, 2009a) who used the philosophy of Dyson and a hydrodynamical description hidden underneath to arrive at the celebrated Burgers' equation:

$$\partial_\tau G(z;\tau) + G(z;\tau)\partial_z G(z;\tau) = 0, \tag{1.7}$$

for the macroscopic Green's function $G(z;\tau)$ of (1.3). As this object is encoding the positions of eigenvalues and its evolution is governed by an Euler equation of hydrodynamics, we call it the hydrodynamical description with eigenvalues taking the role of the fluid. This equation was well-known before by physicists and mathematicians alike, the interesting observation is therefore the way of arriving at the solution of (1.7) by the method of (complex) characteristics. It is known to have a certain feature — solutions are defined on the (z,τ) plane and there are lines in this space-time where the solution ceases to be unique. The places where it starts to happen are called the shock lines. In the context of random matrices, these exactly coincide with the edges of the underlying spectrum. The important insight therefore is that the edges of the spectra are also the places where shocks develop in the hydrodynamical picture. Moreover, the endpoints of the support are of interest by two reasons — as simply being distinguished from the eigenvalue sea and because of the new universality classes showing up near these points.

In Chapter 2.3 we describe the first paper (Blaizot et al., 2015a) as a continuation of the programme sketched above. We obtain diffusion type equations for the averaged characteristic polynomial U_N and the averaged inverse characteristic polynomial E_N in the dynamical Gaussian Unitary Ensemble (dGUE):

$$\begin{aligned} \partial_\tau U_N(z;\tau) &= -\frac{1}{2N}\partial_{zz} U_N(z;\tau), \\ \partial_\tau E_N(z;\tau) &= \frac{1}{2N}\partial_{zz} E_N(z;\tau). \end{aligned} \tag{1.8}$$

These averages are connected to the Burgers' equation (1.7) via a Cole–Hopf transform (1.6). They are exact equations valid for any matrix size N, unlike (1.7) where the limit of $N \to \infty$ was already taken. We have obtained the equations (1.8) for arbitrary initial conditions. For completeness we posed the question of universality for both U_N and E_N near the spectral edge and when two such edges collide. We recreated the well-known functions of Airy and Pearcey type for two aforementioned scenarios.

ensemble	equation	definitions
GβE	$\partial_\tau G + G\partial_z G = 0$	G def. in (2.64)
chGβE	$\partial_\tau G^{\mathrm{c}} - \frac{\hat{a}}{z^2} G^{\mathrm{c}} + (G^{\mathrm{c}} + \hat{a}/z)\partial_z G^{\mathrm{c}} = 0$	G^{c} def. in (2.77)
Wishart	$\frac{\hat{a}+1}{2}\partial_\tau G^{\mathrm{W}} + (\hat{a} + 2zG^{\mathrm{W}})\partial_z G^{\mathrm{W}} + (G^{\mathrm{W}})^2 = 0$	G^{W} def. in (2.92)
CβE	$\partial_\tau G^{\circ} + G^{\circ}\partial_z G^{\circ} = 0$	G° def. in (2.99)
Jacobi	$\partial_\tau G^{\mathrm{J}} + \left(\frac{\hat{a}}{2}\cot z/2 + G^{\mathrm{J}}\right)\partial_z G^{\mathrm{J}} - \frac{\hat{a}}{4}\frac{G^{\mathrm{J}}}{\sin^2 z/2} = 0$	G^{J} def. in (2.108)

TABLE 1.1: Burgers' equations of Gaussian, Chiral, Wishart, Circular and Jacobi dynamical matrix models. The parameter $\hat{a} \equiv \lim_{n,m\to\infty} \frac{n}{m} - 1$ with n, m defined separately for each model.

Moreover, these two observables fix completely the correlation kernel of the dynamical Gaussian Unitary Ensemble and thus serve as the models' basic building blocks.

In Chapter 2.5 we discuss the publication (Forrester and Grela, 2016) which looked into the fluid-like description of random matrices from two different but equivalent perspectives — the approach of Dyson and the approach of collective variables. We have arranged in Tab. 1.1 the results of the study of Gaussian, Circular, Chiral, Wishart and Jacobi dynamical models to confirm the robustness of Burgers' equation in the random matrix context.

Additionally, by the method of (complex) characteristics we re-derive compact solution formulae for the initial value problems for equations listed in Tab. 1.1. In the same work we use the collective variables approach to find the asymptotic behaviour of the Harish-Chandra/Itzykson–Zuber and Berezin–Karpelevich integrals. The first one is known to show up in the Gaussian Ensembles, the second is present in the Chiral/Wishart models. Although the HC/IZ integral was studied in the large N limit before, the asymptotic form of the latter integral is considered a new result.

In Chapter 2.4 we discuss the last paper (Blaizot et al., 2016) in this part. It is devoted to the stationary limit $t \to \infty$ of the dynamical matrix. Although in the seminal work of Dyson an external quadratic potential (multi-dimensional generalization of the Ornstein–Uhlenbeck process) was introduced to attain a non-zero stationary limit, we show that it is a largely superficial complication. This is shown by the multi-dimensional Lamperti transformation which generally reduces the dynamics of Ornstein–Uhlenbeck type to a free diffusion.

Non-Hermitian dynamical random matrices. Successful application of dynamical Gaussian Unitary Ensemble (and others listed in Tab. 1.1) encouraged to look for the traces of similar hydrodynamical structures in the non-Hermitian realm. After introducing analogous dynamical models, preliminary studies of Chapter 3.1.2 show that the induced eigenvector and eigenvalue dynamics have at least one distinct feature in comparison to the dGUE case — the eigenvalues and eigenvectors are intermingled and cannot be decoupled. Because the robustness of Burgers' equation found in previous examples is traced back to this decoupling, advancing the hydrodynamical picture into the non-Hermitian world is not straightforward and the way to proceed is to introduce an observable including the information on eigenvector. In Chapter 3.2 we discuss on how this task was completed in the work of (Burda et al., 2014) which laid the foundations to a hydrodynamical description akin to (1.7). To this end we form an extended characteristic polynomial $D_N = \left\langle \det\left((z - X)(\bar{z} - X^\dagger) + |w|^2\right)\right\rangle$

and derive a diffusion equation:

$$\partial_\tau D_N(Q;\tau) = \frac{1}{N}\partial_{w\bar{w}} D_N(Q;\tau), \tag{1.9}$$

where $Q = (z, w)$ is a pair of two complex numbers (or a quaternion). It is an exact and novel result which makes the fluid-like description possible. The extended characteristic polynomial has its roots in the electrostatic analogy with w parameter being a regulator of the $\log$ potential. The evolution of (1.9) happens in the auxiliary parameter w space whereas now the z space where the eigenvalues reside is a dummy parameter. This exchange of roles is most profoundly seen in the pair of Burgers' equations:

$$\partial_\tau V = \partial_w |V|^2, \tag{1.10}$$

$$\partial_\tau H = \partial_z |V|^2, \tag{1.11}$$

derived by a Cole–Hopf transform and after taking a large matrix size limit. The transforms H, V are the building blocks of the macroscopic observables — the spectral density (H) and the eigenvector correlator (V) as is discussed in Chapter 3.1.1. Importantly, only the first equation of (1.10) is an independent Burgers' equation whereas the second (1.11) is auxiliary. We conclude that the eigenvector correlator is a deciding object in the evolution of both eigenvalues and eigenvectors in the regime of large matrices. We inspect on how it is not a surprise as non-zero w variables exactly couple with the eigenvector degrees of freedom.

In Chapters 3.2.2 and 3.3 we report on the paper (Burda et al., 2015) in which the equation (1.9) was studied in more detail. Beseides the diffusion (1.9) and Burgers' equations (1.10),(1.11) we inspect an another equivalent equation of Hopf–Lax (obtained by a "half" of the Cole–Hopf transform) which turns out to be most useful in practical calculations. We discuss three examples in which the role of normal and non-normal initial conditions, question of universality and the behaviour of characteristic lines are discussed. Novel formulae for the macroscopic spectral densities and eigenvector correlators read

$$\rho(z;\tau) = \rho^{(\mathrm{n})}(z;\tau) + \rho^{(\mathrm{nn})}(z;\tau), \tag{1.12}$$

$$\rho^{(\mathrm{n})}(z;\tau) = \lim_{N\to\infty} \frac{1}{N\pi} \frac{1}{\mathrm{Tr}\mathcal{M}^{-2}} \det \begin{pmatrix} \mathrm{Tr}(\bar{z} - X_0^\dagger)\mathcal{M}^{-2} & \mathrm{Tr}\mathcal{M}^{-2} r_* \\ -\mathrm{Tr}\mathcal{M}^{-2} r_* & \mathrm{Tr}(z - X_0)\mathcal{M}^{-2} \end{pmatrix},$$

$$\rho^{(\mathrm{nn})}(z;\tau) = \lim_{N\to\infty} \frac{1}{N\pi} \mathrm{Tr}\left(\mathcal{M}^{-1}[\mathcal{M}^{-1}; z - X_0](\bar{z} - X_0^\dagger)\right),$$

$$O(z;\tau) = \frac{1}{\pi\tau^2} r_*^2, \tag{1.13}$$

with $\mathcal{M}(z,r) \equiv (z - X_0)(\bar{z} - X_0^\dagger) + r^2$ and condition for the r_* given by

$$\lim_{N\to\infty} \frac{1}{N} \mathrm{Tr}\mathcal{M}(z, r_*)^{-1} = \frac{1}{\tau}. \tag{1.14}$$

The distinction between normal and non-normal initial matrices X_0 is most transparent in this form as $\rho^{\mathrm{nn}} = 0$ if X_0 is normal.

In a short Chapter 3.4 we revisit the work (Blaizot et al., 2016) and discuss the Lamperti transformation in the non-Hermitian setting. The conclusions are similar as before and also in this case there is no essential need of a confining potential.

Lastly, in Chapter 3.5 we address the question of spectral densities in the finite matrix size. We report on the work (Grela and Guhr, 2016) where we consider statistical models of the form

$$M = S + LXR. \tag{1.15}$$

The matrices S, L and R are fixed and represent the structural part whereas X is a stochastic part. For a special choice of $LR = \sqrt{\tau}$ and by setting $S = X_0$ this family reduces with the dynamical Ginibre Unitary Ensembles but otherwise it is more general. In particular, by supersymmetric methods we derive a finite matrix size counterpart of (1.12). We inspect both $L = R = 1$ and $L, R \neq 1$, look into non-normal initial matrices of finite rank and obtain results for the spectrum of M^{-1}. All of these results are considered new.

Diffusion method. The Chapter 4 is devoted to a computational tool formalized in (Grela, 2016) and used throughout this thesis to obtain diffusive equations of the type (1.8) and (1.9). The purpose of the method is to calculate observables containing averaged characteristic determinants (products or ratios). The main idea is that highly-dimensional (of order N^2) dynamics of matrix elements induce a lower dimensional (i.e. of order k^2 where k is the number of characteristic determinants) diffusion equation for the observable in questions. The variables in which dual diffusion takes place remain to be identified and is a part of the method — in the case of characteristic polynomials U_N, E_N it was simply their argument z whereas in the non-Hermitian setting, a new (dual-) variable w was introduced. We apply this technique to a couple of interesting examples — we compute the averaged ratio of determinants of Hermitian (dGUE) random matrices by which we re-obtain the correlation kernel, we consider products of k extended characteristic polynomials in the dynamical Ginibre Ensemble and we study the crossover model between the real and complex Ginibre random matrix. In the case of products of k characteristic polynomials, a duality formula was found whereas in the crossover model we could probe on how a comb of real eigenvalues is developed as we vary the parameter β between $\beta = 1$ and $\beta = 2$. The second and third examples were not considered in the literature before whereas the first is complementary to the study discussed in Chapter 2.3.

Chapter 2

Dynamical random matrices

2.1 Stochastic processes

Since throughout this dissertation we view dynamical random matrices as stochastic processes, we now introduce basic concepts of stochastic analysis (Gillespie, 1992; Kampen, 2007). For clarity's sake we restrict to just one dimensional problem and generalize the framework in the end. A stochastic process X_t is defined as a random function of a real time-like parameter t and random variable X drawn from a given probability density function $p_X(y)dy$. In this work we deal with two processes of Wiener and Ornstein–Uhlenbeck type applied to the random matrices. They both satisfy the Markov property i.e. are memoryless and therefore the conditional probabilities satisfy:

$$P_{1|n-1}\left(x_n, t_n | x_1, t_1; x_2, t_2; ...; x_{n-1}, t_{n-1}\right) = P_{1|1}\left(x_n, t_n | x_{n-1}, t_{n-1}\right), \tag{2.1}$$

where $P_{n|m}\left(x_{m+1}, t_{m+1}; ...; x_{m+n}, t_{m+n} | x_1, t_1; ...; x_m, t_m\right)$ is the probability of finding $x_{m+1}...x_{m+n}$ at times $t_{m+1}...t_{m+n}$ given the points $x_1...x_m$ at times $t_1...t_m$ with chronological order assumed. Intuitively, the condition (2.1) is simple — the probability of arriving at the point x_n, t_n depends only on x_{n-1}, t_{n-1}, the immediate past of the process. It is a very strong constraint on the joint probability density functions and so just two functions P_1 (one-point probability density function) and $P_{1|1}$ (transition function) characterize any Markov process completely. Additionally, transition functions necessarily obey the Chapman–Kolmogorov equations:

$$P_{1|1}(x_1, t_1 | x_3, t_3) = \int P_{1|1}(x_1, t_1 | x_2, t_2) P_{1|1}(x_2, t_2 | x_3, t_3) dx_2, \tag{2.2}$$

with $t_1 \geq t_2 \geq t_3$ and likewise for P_1 we find

$$P_1(x_2, t_2) = \int P_{1|1}(x_2, t_2 | x_1, t_1) P_1(x_1, t_1) dx_1. \tag{2.3}$$

One can also argue the opposite — any two positive probabilistic functions $P_1, P_{1|1}$ obeying (2.2) and (2.3) describe a Markov stochastic process. Thus, we can simply define the Wiener process by two such functions

$$\begin{aligned} P_1(x, t = 0) &= \delta(x), \\ P_{1|1}(x_2, t_2 | x_1, t_1) &= \frac{1}{\sqrt{2\pi\delta t}} \exp\left(-\frac{(x_2 - x_1)^2}{2\delta t}\right), \quad \delta t \geq 0, \end{aligned} \tag{2.4}$$

where $\delta t = t_2 - t_1$ and $P_1(x, t)$ is computed from (2.3) and the fact that $P_{1|1}$ depends only on δt. In physics such processes were considered by Einstein (Einstein, 1905)

and Smoluchowski (Smoluchowski, 1906) among others to describe the motion of microscopic particles in a "stochastic bath" comprised of smaller corpuscles. There are however two other forefathers worth mentioning — Bachelier (Bachelier, 1900) whose study was pioneering application of stochastic processes in finances and Wiener who was the first to study it as a purely mathematical problem.

The functions (2.4) describe also a diffusive mechanism, as will be shown later on. The Ornstein–Uhlenbeck process is in turn completely characterized by

$$\begin{aligned} P_1(x, t=0) &= \delta(x), \\ P_{1|1}(x_2, t_2|x_1, t_1) &= \frac{1}{\sqrt{2\pi(1-\kappa^2)}} \exp\left(-\frac{(x_2 - \kappa x_1)^2}{2(1-\kappa^2)}\right), \end{aligned} \tag{2.5}$$

where $\kappa = e^{-\frac{1}{2}\delta t}$. These provide a classic example of a stationary process which has a non-zero limiting distribution $P_1(x,t) \overset{t\to\infty}{\to} (2\pi)^{-1/2} \exp\left(-\frac{1}{2}x^2\right)$. We will show in Chapter 2.4 how (2.11) and (2.5) can be related by a Lamperti transformation.

Although at this point we are able to calculate averages, a partial differential equation describing the stochastic behaviour will prove to be both useful and familiar. We consider $P_{1|1}(x_2, t_2|x_1, t_1) \equiv T_{\delta t}(x_2|x_1)$ for small time interval δt wherein only one transition takes place. If so, two possibilities can occur — no transition happens ($x_2 = x_1$) with rate $(1 - a_1\delta t)$ or a transition $W(x_2|x_1)$ takes place $x_1 \to x_2$ with rate δt. The constant reads $a_1(x_1) = \int dx_2 W(x_2|x_1)$ as it is the probability of making any transition $x_1 \to x_2$. We have thus an approximate formula

$$T_{\delta t}(x_2|x_1) = (1 - a_1(x_1)\delta t)\delta(x_2 - x_1) + \delta t W(x_2|x_1) + O(\delta t^2), \tag{2.6}$$

with $T_{\delta t} \to \delta(x_2 - x_1)$ as $\delta t \to 0$, in accordance with our intuitions. We plug it into the Chapman–Kolmogorov formula (2.2) and in the limit $\delta t' \to 0$ arrive at the equation

$$\partial_t T_t(y|x) = \int dy' \left[T_t(y'|x)W(y|y') - T_t(y|x)W(y'|y)\right], \tag{2.7}$$

which is the Master equation for the transition function T_t (the conditional probability $P_{1|1}$) given unknown $W(x|y)$. It is, as its name bears, a powerful formula and we do not discuss its importance at all — it serves merely as an intermediate step in what follows. As a last step, we expand the first term of (2.7) around $y' = y - r$:

$$T_t(y'|x)W(y|y') = \sum_{n=0}^{\infty} \frac{(-r)^n}{n!} \frac{\partial^n}{\partial y^n} \left[T_t(y|x)W(y+r|y)\right], \tag{2.8}$$

and obtain an infinite order partial differential equation called the Kramers–Moyal equation:

$$\partial_t T_t(y|x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial y^n} \left[T_t(y|x)a_n(y)\right], \tag{2.9}$$

where the transition moments read $a_n(y) = \int dr W(y+r|y)r^n$. When only two first moments a_1, a_2 are dominant and non-zero, the Smoluchowski–Fokker–Planck equation (hereafter SFP) finally arise:

$$\partial_t P_t(y|x) = \frac{1}{2}\partial_{yy}\left[a_2(y)P_t(y|x)\right] - \partial_y\left[a_1(y)P_t(y|x)\right], \tag{2.10}$$

where we reintroduced the notation resembling the probability density function $T_t(y|x) \equiv P_t(y|x)$ and an initial condition is given by $P_0(y|x)$. We now turn back to (2.4) and (2.5) where two stochastic processes were defined by the initial function P_1 and the transition function $P_{1|1}$. In the language of SFP equation, the latter function is the solution of (2.10) with an initial condition given by the former. Therefore, the Wiener process of (2.11) is a solution with $a_1 = 0$ and $a_2 = 1$ whereas the Ornstein–Uhlenbeck process of (2.5) is realized for $a_1 = -y/2$ and $a_2 = 1$. These values characterize the process fully — Wiener motion is thus a free diffusion due to non-zero and constant a_2 whereas the Ornstein–Uhlenbeck motion is diffusive (hence the same value of a_2) although with a quadratic external potential applied (hence a non-zero linear force term a_1).

Lastly, we turn to a microscopic approach describing the stochastic process as a Brownian motion. So far we have constructed a family of probability density functions (2.1) and worked on a "smoothened" ensemble-oriented problem. Sometimes it is however advantageous to study the stochastic process X_t itself as a single trajectory of a rough and highly unpredictable nature (in fact, nowhere differentiable). After all, exactly such trajectories were observed by Brown in pollen grains surrounded by water molecules. Here we show how to simulate such motion. We formulate dynamics of the form:

$$X_{t+\delta t} = X_t + \delta X, \tag{2.11}$$

with small increment δt and stochastic part δX drawn from a normal distribution with two first moments non-zero. It is simply describable:

$$\begin{aligned} \langle \delta X \rangle &= a_1(X_t)\delta t, \\ \left\langle (\delta X)^2 \right\rangle &= a_2(X_t)\delta t. \end{aligned} \tag{2.12}$$

An ensemble of trajectories created this way and for suitably chosen parameters a_1 and a_2 is likewise describable by probabilities (2.4) and (2.5). Eventually, a prescription (2.11) is an equivalent form defining both Wiener and Ornstein–Uhlenbeck processes.

We have so far introduced two equivalent frameworks describing stochastic processes — SFP equation and Brownian motion and now we turn to the discussion of matrices undergoing diffusive dynamics.

2.2 Dynamical matrices

We look at stochastic (or dynamical) matrices M_t as multi-dimensional generalizations of processes X_t. As a Brownian motion, consider a rectangular matrix $(M_t)_{ij}$ of size $N \times \tilde{N}$ dependent on a time parameter t with the dynamics defined by:

$$M_{t+\delta t} = M_t + \delta M, \tag{2.13}$$

where the increment matrix δM is assumed to be Gaussian and thus defined by its first two moments:

$$\begin{aligned} \langle \delta M_{ij} \rangle &= (a_1)_{ij}\delta t, \\ \langle \delta M_{ij} \delta M_{kl} \rangle &= (a_2)_{ij,kl}\delta t, \end{aligned} \tag{2.14}$$

β	Δ_M operator	Moments	Matrix form	Symmetries
1	$\sum_{i=1}^{N} \partial^2_{x_{ii}} + \frac{1}{2} \sum_{i<j=1}^{N} \partial^2_{x_{ij}}$	$\langle \delta M_{ij} \rangle = 0$ $\langle \delta M_{ij}^2 \rangle = g_{ij} \delta t$	$M_{kl} = x_{kl}$	$x_{kl} = x_{lk}$
2	$\frac{1}{2} \sum_{i=1}^{N} \partial^2_{x^{(1)}_{ii}} + \frac{1}{4} \sum_{d=1}^{2} \sum_{i<j=1}^{N} \partial^2_{x^{(d)}_{ij}}$	$\langle \delta M_{ij} \rangle = 0$ $\langle \lvert \delta M_{ij} \rvert^2 \rangle = \delta t$	$M_{kl} = x^{(1)}_{kl} + i x^{(2)}_{kl}$	$x^{(1)}_{kl} = x^{(1)}_{lk}$ $x^{(2)}_{kl} = -x^{(2)}_{lk}$
4	$\frac{1}{4} \sum_{i=1}^{N} \partial^2_{x^{(1)}_{ii}} + \frac{1}{8} \sum_{d=1}^{4} \sum_{i<j=1}^{N} \partial^2_{x^{(d)}_{ij}}$	$\langle \delta M_{ij} \rangle = 0$ $\langle \lvert \delta M_{ij} \rvert^2_Q \rangle = \frac{\delta t}{g_{ij}}$	$M_{kl} = \sum_{d=1}^{4} x^{(d)}_{kl} \tilde{\sigma}_d$	$x^{(1)}_{kl} = x^{(1)}_{lk}$ $x^{(2)}_{kl} = -x^{(2)}_{lk}$ $x^{(3)}_{kl} = -x^{(3)}_{lk}$ $x^{(4)}_{kl} = -x^{(4)}_{lk}$

TABLE 2.1: Spatial operators Δ_M of (2.16) and moments of stochastic increments of (2.13) for Wiener (i.e. free) dynamical matrices. Real and complex matrices are N dimensional, symplectic case is of size $2N$. The $\tilde{\sigma}_d$'s form a four-vector of 2×2 matrices $(1_2, i\sigma_3, i\sigma_2, i\sigma_1)$ where σ_i are Pauli matrices and $|x|^2_Q$ is the quaternionic norm.

which is a generalization of (2.12). To arrive at the multi-dimensional SFP equation (2.10), we introduce a joint probability density function (hereafter jPDF):

$$P_t\left(M|M_0\right) d[M] \equiv \prod_{i,j \in \mathrm{dof}(M)} P_t\left(M_{ij}|(M_0)_{ij}\right) dM_{ij}, \tag{2.15}$$

where the notation $\mathrm{dof}(M)$ enumerate the degrees of freedom of M. The multi-dimensional SFP equation has a general structure:

$$\begin{aligned} \partial_t P_t &= \frac{1}{2} \sum_{i,j,k,l \in \mathrm{dof}(M)} \partial_{M_{ij},M_{kl}} \left[(a_2)_{ij,kl} P_t\right] - \sum_{i,j \in \mathrm{dof}(M)} \partial_{M_{ij}} \left[(a_1)_{ij} P_t\right] \equiv \\ &\equiv \Delta_M P_t, \end{aligned} \tag{2.16}$$

where Δ_M is a multi-dimensional operator generalizing the r.h.s. of (2.10) and we we suppressed the arguments $P_t = P_t\left(M|M_0\right)$. It is reinforced with initial condition of the form $P_0\left(M|M_0\right) = \prod_{i,j \in \mathrm{dof}(M)} \delta\left(M_{ij} - (M_0)_{ij}\right)$ with a fixed matrix M_0. For concreteness, in this section we restrict to a classic trio of dynamical Gaussian Orthogonal ($\beta = 1$), Unitary ($\beta = 2$) and Symplectic ($\beta = 4$) Ensembles (hereafter dGOE, dGUE and dGSE or altogether as dGβE). We set $N = \tilde{N}$ and in Tab. 2.1 we present the operators Δ_M, matrix form and the symmetries of these three families. All three dGβE's are the matrix analogues of the Wiener processes for symmetric, Hermitian and self-dual matrices respectively. The Ornstein–Uhlenbeck type of evolution is also possible and will be addressed in Sec. 2.4.

At this point we see clearly the justification for introducing the dynamics. Multi-dimensional equation (2.16) is fully separable for each degree of freedom, linear and of second order — an initial value problem can be solved readily for each β:

$$P_t^{\beta}\left(M|M_0\right) = \frac{1}{C^{\beta}_{N,t}} \exp\left(-\frac{\beta}{4t} \mathrm{Tr}\left(M - M_0\right)^2\right), \tag{2.17}$$

with proper normalization constants $C^{\beta}_{N,t}$.

For concreteness we consider the dGUE ($\beta = 2$) case when $M \equiv H$ is Hermitian. In

particular case of $H_0 = 0$ we recreate the well-known jPDF of GUE $\sim \exp(-\mathrm{Tr}H^2)$. For non-zero H_0, the (equal-time) dynamical matrix is expressible as a static model with a fixed matrix H_0 (sometimes called an external source) and variance proportional to time parameter t. We have thus followed the route of Dyson although in the opposite direction — the question posed in his seminal paper (Dyson, 1962a) was the following — since I know the jPDF of eigenvalues for GβE, is there any dynamical model dGβE which reproduces the same densities? As we know now, the answer is positive and a proper model of (2.16) is stochastic in nature. Furthermore, he took the classic problems of random matrix theory and re-expressed them in a dynamical framework — in particular, the study of dynamics of eigenvalues and eigenvectors under the stochastic motion of (2.16).

At this point a fairly technical problem arises, all processes defined in Tab. 2.1 are free — they do not have a stationary limit as $t \to \infty$. Hence at first the relation between the dynamics and statics is lost as Dyson retrieves the jPDF of GβE only in that limit. However, when looking at the matrix models of (2.17), we observe that only a time-dependent variance is the trace of the underlying dynamics. Therefore, at an every snapshot of t, the dynamical model and a "frozen" matrix model coincide. Moreover, it is possible to include a stationary limit by a Lamperti transformation which was discussed in Chapter 2.4.

2.2.1 Dyson's idea of eigenvalue and eigenvector dynamics

We discuss a natural question when dynamical matrices are studied — the Brownian motion on each matrix element must induce a certain dynamics on both eigenvalues and eigenvectors. What is its form? In static models of GβE we start from (2.17) (with t dependent variance and $\beta = 2$), diagonalize $H = U\Lambda U^\dagger$ and integrate over the angular (eigenvector) variables U which introduces the Vandermonde repulsion between the eigenvalues Λ. Dynamical picture is similar and retrieving the eigenvalue behaviour can proceed in two ways — we can start from the SFP equation of (2.16) and compute the operator Δ_H in new variables $H \to (U, \Lambda)$. We arrive at a multidimensional equation with both Λ and U terms present. It has however an important property — it factorizes quite miraculously and so we can integrate over the angular variables. This is exactly the dynamical counterpart of the Harish-Chandra/Itzykson–Zuber integral found in static models. Another approach of Dyson is based on the perturbation formulae and is both simpler and more intuitive from the physics point of view — we review it here for the dGUE case. Consider the eigenvalue problem:

$$H_{t'} \left|\psi_i(t')\right\rangle = \lambda_i(t') \left|\psi_i(t')\right\rangle , \tag{2.18}$$

with $t' = t + \delta t$. By (2.13) it forms a typical perturbation problem of quantum mechanics when δH is small. The eigenvalue corrections read $\delta\lambda_i \equiv \lambda_i(t') - \lambda_i(t)$ and are found by standard techniques in the Appendix A:

$$\delta\lambda_i = \delta H_{ii} + \sum_{k(\neq i)} \frac{\delta H_{ki}\delta H_{ik}}{\lambda_i - \lambda_k} + ..., \tag{2.19}$$

with $\delta H_{ij} \equiv \langle\psi_i(t)|\delta H|\psi_j(t)\rangle$ and $\lambda_i \equiv \lambda_i(t)$ and higher-order terms are skipped. We remind of a classic quantum-mechanical interpretation of the above formula — eigenvalues typically repel each other when we perturb the systems' Hamiltonian. The Vandermonde interaction is therefore this precise phenomenon in the random matrix

realm and is a serious hint on why certain quantum-mechanical problems are successfully addressed by the random matrix theory. Averaging (2.19) over δH according to Tab. 2.1 produces moment formulae:

$$\begin{aligned}\langle \delta\lambda_i \rangle &= \delta t \sum_{k(\neq i)} \frac{1}{\lambda_i - \lambda_k}, \\ \langle \delta\lambda_i \delta\lambda_j \rangle &= \delta_{ij}\delta t.\end{aligned} \tag{2.20}$$

which, by equations (2.12) are equivalent to the SFP equation:

$$\partial_t P_t = \frac{1}{2}\sum_i \partial^2_{\lambda_i} P_t - \sum_{i\neq k} \partial_{\lambda_i}\left(\frac{1}{\lambda_i - \lambda_k} P_t\right), \tag{2.21}$$

where the arguments of $P_t = P_t(\lambda|\lambda_0)$ were suppressed. Equivalently, we form a Stochastic Differential Equation (hereafter SDE):

$$d\lambda_i = \sum_{k(\neq i)} \frac{dt}{\lambda_i - \lambda_k} + dW_i, \tag{2.22}$$

with a standard Wiener process defined by the second moment $dW_i dW_j = \delta_{ij} dt$. In a similar fashion elucidated in the Appendix A, we derive the deviation for the eigenvectors $\delta\,|\psi_i\rangle \equiv |\psi_i(t')\rangle - |\psi_i(t)\rangle$ as

$$\begin{aligned}\delta\,|\psi_i\rangle = &\sum_{k(\neq i)} \frac{\delta H_{ki}}{\lambda_i - \lambda_k}|\psi_k\rangle + \sum_{l,k(\neq i)} \frac{\delta H_{ki}\delta H_{lk}}{(\lambda_i - \lambda_k)(\lambda_i - \lambda_l)}|\psi_l\rangle + \\ &- \sum_{l(\neq i)} \frac{\delta H_{ii}\delta H_{li}}{(\lambda_i - \lambda_l)^2}|\psi_l\rangle - \frac{1}{2}\sum_{k(\neq i)} \frac{\delta H_{ki}\delta H_{ik}}{(\lambda_i - \lambda_k)^2}|\psi_i\rangle\,.\end{aligned} \tag{2.23}$$

We introduce $|\psi_k\rangle_j \equiv U^{(x)}_{kj} + iU^{(y)}_{kj}$ and compute the first two moments:

$$\begin{aligned}\left\langle \delta U^{(\sigma)}_{in} \right\rangle &= -\frac{\delta t}{2}\sum_{l(\neq i)} \frac{U^{(\sigma)}_{in}}{(\lambda_i - \lambda_l)^2}, \\ \left\langle \delta U^{(\sigma)}_{in}\delta U^{(\sigma')}_{jm} \right\rangle &= \delta t\delta_{ij}\sum_{k(\neq i)} \frac{U^{(\sigma)}_{kn}U^{(\sigma')}_{km}}{(\lambda_i - \lambda_k)^2},\end{aligned} \tag{2.24}$$

with $\sigma, \sigma' = \{x, y\}$. We find expressing the eigenvector dynamics as a SDE to be the most succinct:

$$d\,|\psi_i\rangle = -\frac{1}{2}\sum_{k(\neq i)} \frac{dt}{(\lambda_i - \lambda_k)^2}|\psi_i\rangle + \sum_{k(\neq i)} \frac{dW_{ik}}{\lambda_i - \lambda_k}|\psi_k\rangle\,, \tag{2.25}$$

with a real Wiener matrix process defined by the second moment $dW_{ik}dW_{jl} = \delta_{ij}\delta_{kl}dt$. The picture is now complete — out of the dynamics of (2.16), the eigenvalues and eigenvectors evolve according to (2.22) and (2.25). Importantly, even though the equation (2.25) depends on the eigenvalues λ_i, the influence is not mutual — no eigenvectors are present in the dynamics of eigenvalues themselves (2.22). This makes the dynamics of Hermitian matrices relatively simple — the eigenvalues are the independent degrees of freedom whereas the elements $|\psi_k\rangle$ evolve only subjected to the (pre-determined) λ_i trajectories. We conclude that the eigenvalues and eigenvectors

of dGUE are however largely decoupled from each other. Such stochastic processes were studied in detail by (Allez, Bun, and Bouchaud, 2014).

2.3 Characteristic polynomials in free dynamical GUE

After introducing the dynamical models dGβE in Tab. 2.1, we turn to the study of characteristic polynomials in the particular case of dGUE. In this part we rescale the time variable $Nt = \tau$ which addresses the presence of two time scales which arise in dynamical models as noticed in (Dyson, 1962a): the macroscopic time-scale $t \sim 1$ on which the overall shape of eigenvalue gas is formed and a microscopic scale $t \sim 1/N$ on which the eigenvalue repulsion happens. We define two objects — the averaged characteristic polynomial (hereafter ACP) and the averaged inverse characteristic polynomial (hereafter AICP):

$$U_N(z;\tau) \equiv \langle \det(z - H_\tau) \rangle , \qquad E_N(z;\tau) \equiv \left\langle \det(z - H_\tau)^{-1} \right\rangle , \tag{2.26}$$

which were studied initially by Brézin and Hikami in their papers (Brézin and Hikami, 2000; Brézin and Hikami, 2001). They encode the eigenvalue information and turn out to be both simple and fundamental. We relate the ACP and the Green's function in the the large N limit as:

$$\lim_{N\to\infty} \frac{1}{N} \partial_z \log U_N(z;\tau) = \lim_{N\to\infty} G_N(z;\tau) \equiv G(z;\tau), \tag{2.27}$$

by using the self-averaging property $\lim_{N\to\infty} \log \langle f \rangle = \lim_{N\to\infty} \langle \log f \rangle$ and with $G_N(z;\tau)$ defined in (1.3). Additionally, they form the building blocks of the correlation kernel function. More generally, the averaged ratios of characteristic determinants were studied extensively both for its own sake (Fyodorov and Strahov, 2003; Borodin and Strahov, 2006) and as a practical quantity to obtain the eigenvalue correlation functions (Guhr, 1991). In Chapter 4.2.1 we computed the averaged ratio of determinants in dGUE as an example of the diffusion method.

2.3.1 Solutions, asymptotics and the kernel

In the paper (Blaizot et al., 2015a) we study both the ACP and the AICP in the dGUE model defined by the SFP equation (2.16) and the second row of Tab. 2.1. We present two simple partial differential equations for both U_N and E_N:

$$\partial_\tau U_N(z;\tau) = -\frac{1}{2N} \partial_{zz} U_N(z;\tau), \tag{2.28}$$

$$\partial_\tau E_N(z;\tau) = \frac{1}{2N} \partial_{zz} E_N(z;\tau), \tag{2.29}$$

valid for any initial condition H_0 and for derivation we relegate to the source paper (Blaizot et al., 2015a). It is a continuation of the study (Blaizot and Nowak, 2010) where the equations (2.28) and (2.29) were shown to hold only for the trivial initial condition $H_0 = 0$. In this work we additionally address the universal large N limit of U_N and E_N in the Airy and Pearcey regimes and reconstruct the correlation kernel for the dynamical GUE.

The main advantage of the equations (2.28) and (2.29) is that they have obvious integral solutions dependent on the initial conditions. In this section we explicitly state those representations. Let us also note here that these types of integrals were obtained

(Bleher and Kuijlaars, 2005) as representations of multiple orthogonal polynomials (Desrosiers and Forrester, 2008; Bleher and Kuijlaars, 2004b) and equivalently as an averaged characteristic polynomials of GUE matrices perturbed by an external source (Forrester, 2013). One can verify by a direct calculation that the expression for the ACP reads

$$U_N(z;\tau) = \sqrt{\frac{N}{2\pi\tau}} \int_{-\infty}^{\infty} \exp\left(-N\frac{(q-iz)^2}{2\tau}\right) \prod_{i=1}^{N}(-iq-\lambda_i^0)\, dq. \tag{2.30}$$

where the rotated argument iq arises as a consequence of the negative value of the diffusion constant in the equation (2.28). The most general form of the initial condition is diagonal $H_0 = \mathrm{diag}(\lambda_1^0, ..., \lambda_N^0)$.

The integral representation of the AICP arising as a solution to the partial differential equation (2.29) reads

$$E_N^{\pm}(z;\tau) = \sqrt{\frac{N}{2\pi\tau}} \int_{\Gamma_\pm} \exp\left(-N\frac{(q-z)^2}{2\tau}\right) \prod_{i=1}^{N}(q-\lambda_i^0)^{-1} dq. \tag{2.31}$$

where two possible contours $\Gamma_\pm$ are defined as avoiding all of the poles either from above (Γ_+) or below (Γ_-) the real axis. By additionally imposing that the solutions $E_N^{\pm}$ coincide with the initial conditions as $\tau \to 0$, by a saddle point analysis we find that E^+ (E^-) satisfies the initial condition when $\mathrm{Im} z > 0$ ($\mathrm{Im} z < 0$). The overall solution to (2.29) is thus defined on $z \in \mathbb{C} \setminus \mathbb{R}$ as:

$$E_N(z;\tau) = \begin{cases} E_N^+(z;\tau), & \mathrm{Im} z > 0, \\ E_N^-(z;\tau), & \mathrm{Im} z < 0. \end{cases} \tag{2.32}$$

As the formulae (2.30) and (2.31) are exact for finite N, we report on a handful of asymptotic calculations as $N \to \infty$. We utilize a standard saddle-point/steepest descent analysis(Wong, 2001) of integrals of the form

$$\int_\Gamma e^{Nf(p,z,\tau)} dp, \tag{2.33}$$

along a contour Γ and for a certain function f determining the saddle-points.

Firstly, for the simplest initial condition of a null matrix $H_0 = 0$, we re-obtain the well-known Airy functions. For the ACP case, we read-off the function f as $f(q,z,\tau) = \ln q - \frac{1}{2\tau}(q-iz)^2$ and the saddle point equation reads $\tau = q(q-iz)$. We are interested in the vicinity of $z_c = 2\sqrt{\tau}$ where the merging of saddle points at a position $q_c = i\sqrt{\tau}$ happens. We inspect the vicinity of z_c (q_c) as $\eta = (z - 2\sqrt{\tau})N^{2/3}$ and $t = (q - i\sqrt{\tau})N^{1/3}$. Expanding the logarithm and taking the limit $N \to \infty$ yields

$$U_N\left(2\sqrt{\tau} + \eta N^{-2/3}; \tau\right) \sim \tau^{N/2} \frac{N^{1/6}}{\sqrt{2\pi}} \exp\left(\frac{N}{2} + \frac{\eta N^{1/3}}{\sqrt{\tau}}\right) \mathrm{Ai}\left(\frac{\eta}{\sqrt{\tau}}\right), \tag{2.34}$$

where

$$\mathrm{Ai}(x) \equiv \int_{\Gamma_0} dt \exp\left(\frac{it^3}{3} + itx\right), \tag{2.35}$$

is the well-known Airy function. The contour Γ_0 is formed by the rays $-\infty \cdot e^{5i\pi/6}$ and $+\infty \cdot e^{i\pi/6}$.

The AICP case is similar — we set $f(u,z,\tau) = -\ln u - \frac{1}{2\tau}(u-z)^2$ and this time the saddle points merge at $u_c = \sqrt{\tau}$. The transformation of variables is given by $\eta = (z - 2\sqrt{\tau})N^{2/3}$ and $it = (u - \sqrt{\tau})N^{1/3}$. The asymptotic results are given again by the Airy function

$$E_N^{\pm}\left(2\sqrt{\tau} + \eta N^{-2/3}; \tau\right) \sim \pm e^{i\phi_\pm} i\tau^{-N/2} \frac{N^{1/6}}{\sqrt{2\pi}} \exp\left(-\frac{N}{2} - \frac{\eta N^{1/3}}{\sqrt{\tau}}\right) \mathrm{Ai}\left(e^{i\phi_\pm}\frac{\eta}{\sqrt{\tau}}\right), \tag{2.36}$$

with phase defined as $\phi_\pm = \pm 2\pi/3$. This is in accordance with previous results for static matrices (Akemann and Fyodorov, 2003).

To probe a different regime, we set up an initial matrix with eigenvalues set to $\pm a$ with equal degeneracy $N/2$ in which case $f = \frac{1}{2}\log(a^2+q^2) - \frac{1}{2\tau}(q-iz)^2$. We determine the saddle point equation and identify an interesting value of parameters near $z_c = 0$ and $\tau_c = a^2$ where all three saddle points merge at $q_c = 0$. To probe the vicinity of this point we set $t = qN^{1/4}, \kappa = (\tau - a^2)N^{1/2}$ and $\eta = zN^{3/4}$. In the limit of $N \to \infty$ we find:

$$U_N\left(\eta N^{-3/4}; a^2 + \kappa N^{-1/2}\right) \sim \frac{N^{1/4}}{\sqrt{2\pi}}(ia)^N \mathrm{P}\left(\frac{\kappa}{2a^2}, \frac{\eta}{a}\right), \tag{2.37}$$

where we have defined the Pearcey integral by:

$$\mathrm{P}(x,y) \equiv \int_{-\infty}^{\infty} dt\ \exp\left(-\frac{t^4}{4} + xt^2 + ity\right). \tag{2.38}$$

In the case of the AICP, the formula reads

$$E_N^{\pm}(z;\tau) = \sqrt{\frac{N}{2\pi\tau}} \int_{\Gamma_\pm} du e^{Nf}, \tag{2.39}$$

with the function $f = -\frac{1}{2}\log(u^2 - a^2) - \frac{1}{2\tau}(u-z)^2$ and $\Gamma_\pm$ denote contours circling the poles at $\pm a$ from above (Γ_+) or from below (Γ_-). We again identify saddle points and find an interesting parameter point where they coalesce and parametrize around it by $\kappa = (\tau - a^2)N^{1/2}$, $\eta = zN^{3/4}$ and $t = e^{-i\pi/4}uN^{1/4}$. We then obtain

$$E_N^{\pm}(\eta N^{-3/4}; a^2 + \kappa N^{-1/2}) \sim \frac{N^{1/4}}{\sqrt{2\pi}}(ia)^{-N} \int_{\tilde{\Gamma}_\pm} dt \exp\left(-t^4/4 - \frac{i\kappa}{2a^2}t^2 + it\frac{e^{-i\pi/4}\eta}{a}\right), \tag{2.40}$$

where a Pearcey type integral is present with the choice of contours $\tilde{\Gamma_\pm}$ depending on the sign of $\mathrm{Im} z$. The $\tilde{\Gamma}_+$ is defined by rays $-\infty e^{i\pi/2}$ and $+\infty$ whereas $\tilde{\Gamma}_-$ is formed by $-\infty$ and $+\infty e^{-i\pi/2}$.

The appearance of a Pearcey type integrals (2.38) in this context is not a surprise — it was found by (Brézin and Hikami, 1996; Zinn-Justin, 1997) when studying the GUE with an external source.

To conclude, we show how the U_N and $E_N^{\pm}$ form the building blocks of the correlation kernel of dynamical GUE. In many previous works (Bleher and Kuijlaars, 2005; Desrosiers and Forrester, 2008; Bleher and Kuijlaars, 2004b; Bleher and Kuijlaars, 2004a), the kernel structure was worked out for an external source model (2.17) for $\beta = 2$. Since the formula (2.17) is also a solution to the SFP equation (2.16), an equivalence between external source model and a dynamical matrix is established.

We can therefore use the kernel results in the former context and reformulate the same results in the latter.

To this end, we set the initial matrix H_0 in a most general form:

$$H_0 = \text{diag}\left(\underbrace{a_1\ a_1\ ...}_{n_1}; \underbrace{a_2\ a_2\ ...}_{n_2}; ...; \underbrace{a_d\ ...}_{n_d}\right), \tag{2.41}$$

with d eigenvalues a_i of multiplicities n_i. Out of the degeneracies we form a multiplicity vector $\vec{n} \equiv (n_1, ..., n_d)$ which has a norm $|\vec{n}| \equiv \sum_{i=1}^{d} n_i = N$ dictated by the matrix size. We subsequently introduce, after (Desrosiers and Forrester, 2008), the multiple orthogonal polynomials of type I and II. The functions of type I are defined through the AICP as:

$$\begin{aligned}\Theta_{\vec{m}}(x;\tau) &\equiv \frac{1}{2\pi i}\left(E^{-}_{|\vec{m}|}(x;\tau) - E^{+}_{|\vec{m}|}(x;\tau)\right) = \\ &= \frac{1}{2\pi i}\sqrt{\frac{N}{2\pi\tau}}\oint_{\Gamma_0} du\ \exp\left(-N\frac{(u-x)^2}{2\tau}\right) E_{\vec{m}}(u;0),\end{aligned} \tag{2.42}$$

with an arbitrary multiplicity vector $\vec{m}$, an initial condition $E_{\vec{m}}(x;0) = \prod_{i=1}^{d}(x-a_i)^{-m_i}$ and the contour Γ_0 encircling all a_i's counter-clockwise. The polynomials of type II are in turn defined through the ACP as:

$$\Pi_{\vec{m}}(x;\tau) \equiv U_{|\vec{m}|}(x;\tau) = \sqrt{\frac{N}{2\pi\tau}}\int_{-\infty}^{\infty} dq\ \exp\left(-N\frac{(q-ix)^2}{2\tau}\right) U_{\vec{m}}(-iq;0), \tag{2.43}$$

with an initial condition $U_{\vec{m}}(x;0) = \prod_{i=1}^{d}(x-a_i)^{m_i}$. We stress the dependency of the polynomials on the multiplicity vector $\vec{m}$ of arbitrary norm $|\vec{m}| \neq N$. As a last step, we introduce an ordering of the vector $\vec{n}$

$$\begin{aligned}\vec{n}^{(0)} &= (0, 0, ..., 0),\\ \vec{n}^{(1)} &= (1, 0, ..., 0),\\ &\vdots\\ \vec{n}^{(n_1)} &= (n_1, 0, ..., 0),\\ \vec{n}^{(n_1+1)} &= (n_1, 1, ..., 0),\\ &\vdots\\ \vec{n}^{(N)} &= (n_1, n_2, ..., n_d).\end{aligned} \tag{2.44}$$

which forms a "nested" sequence ordering the N pairs of type I and type II polynomials

$$\Theta_i(x;\tau) \equiv \Theta_{\vec{n}^{(i+1)}}(x;\tau), \qquad \Pi_i(x;\tau) \equiv \Pi_{\vec{n}^{(i)}}(x;\tau), \qquad i = 0, ..., N-1. \tag{2.45}$$

This stack of functions forms a correlation kernel valid for an arbitrary source H_0

$$K_N(x,y;\tau) = \sum_{i=0}^{N-1}\Theta_i(x;\tau)\Pi_i(y;\tau). \tag{2.46}$$

We plug in the integral representations (2.42) and (2.43) to find

$$K_N(x,y;\tau) = \frac{1}{2\pi i}\frac{N}{2\pi\tau}\oint_{\Gamma_0} du \int_{-\infty}^{\infty} dq e^{-\frac{N}{2\tau}\left((q-iy)^2+(u-x)^2\right)} I(q,u), \tag{2.47}$$

where $I(q,u)$ is the sum over the initial conditions. As an example, we consider the case studied before and we set $a_1 = a, a_2 = -a$ with multiplicities $n_1 = n_2 = N/2$. We calculate the sum $I(q,u)$

$$I(q,u) = \sum_{j=0}^{\frac{N}{2}-1} \frac{(-iq-a)^j}{(u-a)^{j+1}} + \frac{(-iq-a)^{N/2}}{(u-a)^{N/2}} \sum_{j=0}^{\frac{N}{2}-1} \frac{(-iq+a)^j}{(u+a)^{j+1}} =$$
$$= \frac{1}{u+iq}\left(1 - \frac{(-q^2-a^2)^{N/2}}{(u^2-a^2)^{N/2}}\right).$$

and we arrive at the formula given in (Brézin and Hikami, 1996) by noticing that the first term vanishes under the integral:

$$K_N^{(\mathrm{BH})}(x,y;\tau) = \frac{1}{2\pi i}\frac{N}{2\pi\tau}\oint_{\Gamma_0} du \int_{-\infty}^{\infty} dq e^{-\frac{N}{2\tau}\left((q-iy)^2+(u-x)^2\right)} \frac{(-q^2-a^2)^{N/2}}{(u^2-a^2)^{N/2}} \frac{1}{-iq-u}.$$

2.4 Dynamical GUE in a quadratic potential

The dGβE models introduced in Tab. 2.1 (in particular, the dGUE case studied extensively in Chapter 2.3 describe free diffusion — the entries of the matrix are not confined by any potential. Now we turn to discuss a paper (Blaizot et al., 2016) where we introduce a potential to the model of dGUE and thus realize a multi-dimensional Ornstein-Uhlenbeck process of (2.5). We name such models dynamical GβE-OU and to define them we introduce an additional term $\delta\Delta_M$ to the free operator Δ_M listed in Tab. 2.1 so that the full operator in the SFP equation (2.16) reads $\Delta_M^{\mathrm{OU}} \equiv \Delta_M + \delta\Delta_M$. Equivalently, we set first moments of the increments $\langle \delta M_{ij} \rangle$ non-zero and read off the second moments from the Tab. 2.1. Both approaches are collected in Tab. 2.2 which serves as a complementary extension to Tab. 2.1.

β	Additional operator $\delta\Delta_M$	First moment
1	$Na\sum_{i=1}^{N}\partial_{x_{ii}}x_{ii} + Na\sum_{i<j=1}^{N}\partial_{x_{ij}}x_{ij}$	$\langle \delta M_{ij}\rangle = -NaM_{ij}\delta\tau$
2	$Na\sum_{i=1}^{N}\partial_{x_{ii}^{(1)}}x_{ii}^{(1)} + Na\sum_{d=1}^{2}\sum_{i<j=1}^{N}\partial_{x_{ij}^{(d)}}x_{ij}^{(d)}$	$\langle \delta M_{ij}\rangle = -NaM_{ij}\delta\tau$
4	$Na\sum_{i=1}^{N}\partial_{x_{ii}^{(1)}}x^{(1)} + Na\sum_{d=1}^{4}\sum_{i<j=1}^{N}\partial_{x_{ij}^{(d)}}x_{ij}^{(d)}$	$\langle \delta M_{ij}\rangle = -NaM_{ij}\delta\tau$

TABLE 2.2: Additional terms $\delta\Delta_M$ and first moments of stochastic increments defining the models of dGβE confined to a quadratic potential (or dGβE-OU). The spatial operator Δ_M^{OU} of SFP equation (2.16) in these models consists of a confining part $\delta\Delta_M$ and a free part listed in Tab. 2.1. To define dGβE-OU by stochastic moments, the first one is listed in this Table whereas the second is contained in Tab. 2.1.

We again focus on the $\beta = 2$ case as a primary example and we set $M = H$. From the second rows of Tables 2.1 and 2.2 we read off the SFP equation as

$$\partial_\tau P_\tau^a = \sum_{i=1}^{N} \left(\frac{1}{2N} \partial_{x_{ii}}^2 + a \partial_{x_{ii}} x_{ii} \right) P_\tau^a + \sum_{i<j=1}^{N} \left(\frac{1}{4N} \left(\partial_{x_{ij}}^2 + \partial_{y_{ij}}^2 \right) + a \partial_{x_{ij}} x_{ij} + a \partial_{y_{ij}} y_{ij} \right) P_\tau^a. \tag{2.48}$$

where $H_{jk} = x_{jk} + i y_{jk}$, we set $Nt = \tau$ and the arguments of P_τ^a were suppressed $P_\tau^a = P_\tau^a (H|H_0)$. Solution to the initial value problem $P_0^a (H|H_0) = \delta (H - H_0)$ reads:

$$P_\tau^a (H|H_0) = \frac{1}{c_a(\tau)} \exp \left(-\frac{Na}{1 - e^{-2a\tau}} \mathrm{Tr} \left(H - H_0 e^{-a\tau} \right)^2 \right), \tag{2.49}$$

with the constant $c_a(\tau) = \left(\frac{\pi(1-e^{-2a\tau})}{2Na} \right)^{N^2/2} 2^{N/2}$. In the $\tau \to \infty$ limit we recover a static Gaussian Unitary Ensemble with a-dependent variance $P_\tau^a \to \exp \left(-Na \mathrm{Tr} H^2 \right)$. A stationary limit of this type is intuitively expected as any initial ordering of H_0 is lost after a sufficiently long time whereas the confining potential of size $\sim a$ prevents the entries from spreading. On the other hand, as $\tau \to 0$ we obtain $P_\tau^a \to \exp \left(-\frac{N}{2\tau} \mathrm{Tr} \left(H - H_0 \right)^2 \right)$ and conclude that in that limit the process does not feel the external a potential and remembers the initial condition H_0 — it is reduced to a free dynamic model of (2.17) for $\beta = 2$.

Now the task is similar as in the free case — we derive an evolution equation for the averaged characteristic polynomial (2.26). For the details we relegate to the source paper (Blaizot et al., 2016) and here we present only the final result:

$$\partial_\tau U_N(z;\tau) = -\frac{1}{2N} \partial_{zz} U_N(z;\tau) + az \partial_z U_N(z;\tau) - aN U_N(z;\tau). \tag{2.50}$$

Instead of solving it, we change the (z, τ) variables in the following way:

$$\begin{aligned} z' &= e^{a\tau} z, \quad \tau' = \frac{1}{2a} \left(e^{2a\tau} - 1 \right), \\ U_N(z;\tau) &= (1 + 2a\tau')^{-N/2} U'_N(z';\tau'), \end{aligned} \tag{2.51}$$

which is known under the name of Lamperti transformation (derived in Appendix B.1) and reduces equation (2.50) to a free diffusion:

$$\partial_{\tau'} U'_N(z';\tau') = -\frac{1}{2N} \partial_{z'z'} U'_N(z';\tau'). \tag{2.52}$$

It turns out that the model of dGUE-OU (dGUE in a quadratic confining potential) does not produce any new phenomena as we are able to reduce it to the free dGUE studied extensively in Chapter 2.3.

2.5 Matrix hydrodynamics

We present three possible paths to fluid-like behaviour of dynamical random matrices. First stems from the study of averages of characteristic polynomials which were considered in Chapter 2.3, second is a linear response approach to the gas of eigenvalues and third makes use of the collective variables. Last two approaches were discussed

in the work (Forrester and Grela, 2016) as a starting point in establishing the hydrodynamical picture in different dynamical matrix models.

2.5.1 Hydrodynamics from characteristic polynomials

The Cole–Hopf transform (2.27) linking the ACP and the Green's function is a starting point in obtaining the hydrodynamical picture. To this end we introduce a function f_N:

$$f_N(z;\tau) = \frac{1}{N}\partial_z \ln U_N(z;\tau), \tag{2.53}$$

as a Cole–Hopf transform but before taking the $N \to \infty$ limit. Out of (2.28) we retrieve the viscid Burgers' equation:

$$\partial_\tau f_N(z;\tau) + f_N(z;\tau)\partial_z f_N(z;\tau) = -\frac{1}{2N}\partial_{zz} f_N(z;\tau), \tag{2.54}$$

where both non-linear $f_N\partial_z f_N$ and dissipative $\partial_{zz} f_N$ terms are present (Blaizot and Nowak, 2010). The latter is proportional to a viscosity parameter $\nu = -\frac{1}{2N}$ dependent on the matrix size and is sub-leading in the large N limit (also known as the inviscid limit $\nu \to 0$). The inviscid Burgers' equation for f_N becomes an equation for the macroscopic Green's function G since $f_N \overset{N\to\infty}{\longrightarrow} G$:

$$\partial_\tau G(z;\tau) + G(z;\tau)\partial_z G(z;\tau) = 0. \tag{2.55}$$

We stress that both the unknown function G and space-like parameter z are complex and thus (2.55) form a complex variant of the (usually real) Burgers' equation. In Chapter 2.5.2 we describe the method of (complex) characteristics by which first order partial differential equations of the type (2.55) are solved. In the paper (Blaizot et al., 2015a) we discuss how this method is equivalent to a saddle point analysis of the solutions of (2.30) — the method is known to develop shock waves (i.e. places on the (z,τ) plane where the solution ceases to be single-valued) which are directly translated to the merging of the saddle points. Now however we discuss the equation (2.55) in a wider context.

Real Burgers' equation (Burgers, 1974) at first served as a toy model of turbulence containing necessary ingredients of non-linearity and dissipation. Unfortunately, it turned out to be reducible to a heat equation due to a non-linear transformation (2.53) devised independently by Cole and Hopf (Hopf, 1950; Cole, 1951). Despite the failure in understanding turbulence, as a model of different phenomena it still attracts a lot of attention. In particular, we follow it in just the opposite direction — Burgers' equation (2.55) is secondary to the heat equation (2.28) and by this approach the fluid-like dynamics of the resolvent G becomes apparent. Although we describe here only the case of dGUE, we cite papers devoted to the study of dynamical chiral GUE or Wishart models (Blaizot, Nowak, and Warchoł, 2013a; Blaizot, Nowak, and Warchoł, 2013b) and dynamical CUE models (Blaizot and Nowak, 2009a; Blaizot and Nowak, 2008; Neuberger, 2008) from the characteristic polynomials' point of view.

To solve the initial value problem for (2.55), the method of complex characteristics (Marchenko and Pastur, 1967; Voiculescu, Dykema, and Nica, 1992) was invoked where both z and G are complex functions. This is a slight generalization of a standard technique applicable to initial value problems of real first order PDE's. We present this

method for a general first order equation of the form

$$A\Big(G(z;\tau),z,\tau\Big)\partial_\tau G(z;\tau)+B\Big(G(z;\tau),z,\tau\Big)\partial_z G(z;\tau)=C\Big(G(z;\tau),z,\tau\Big). \tag{2.56}$$

The main idea is to seek a coordinate transform $(z,\tau)\to(\alpha,\beta)$ such that the PDE (2.56) becomes an ODE along the curves of constant α:

$$\frac{d}{d\beta}G(z;\tau)=C\Big(G(z;\tau),z,\tau\Big) \tag{2.57}$$

These curves are called characteristic lines or simply characteristics. By the chain rule $\frac{d}{d\beta}=\frac{d\tau}{d\beta}\frac{\partial}{\partial\tau}+\frac{dz}{d\beta}\frac{\partial}{\partial z}$, the left-hand sides of (2.57) and (2.56) dictate the system of equations describing the characteristics

$$\begin{aligned}\frac{d}{d\beta}\tau(\alpha,\beta)&=A\Big(G(\alpha,\beta),z(\alpha,\beta),\tau(\alpha,\beta)\Big),\\ \frac{d}{d\beta}z(\alpha,\beta)&=B\Big(G(\alpha,\beta),z(\alpha,\beta),\tau(\alpha,\beta)\Big),\end{aligned} \tag{2.58}$$

where $G(\alpha,\beta)\equiv G(z(\alpha,\beta);\tau(\alpha,\beta))$. These relations form lines in the (z,τ) space, labelled by the β parameter and passing through the prescribed initial point $(z(\alpha,0),\tau(\alpha,0))$. For the latter to be determined, the Green's function $G(z;\tau=0)$ on the $\tau=0$ line is required. With this initial data specified, the set of equations (2.57) and (2.58) are in principle solvable by standard means and comprise the sought solution to (2.56).

2.5.2 Hydrodynamics by Dysonian approach

The motion of eigenvalues under the diffusive dynamics is described by an SFP equation of a general form (Forrester, 2010):

$$\partial_t P_t\left(\{\lambda\}|\{\lambda^0\}\right)=\mathcal{L}\,P_t\left(\{\lambda\}|\{\lambda^0\}\right),\qquad \mathcal{L}=\sum_{j=1}^{N}\frac{\partial}{\partial\lambda_j}\Big(\frac{\partial W}{\partial\lambda_j}+\beta^{-1}\frac{\partial}{\partial\lambda_j}\Big), \tag{2.59}$$

where $\{x\}$ denote the set of variables x_i and the λ_i^0's are the initial values. The potential W has a general structure:

$$W=\sum_{i=1}^{N}V_1(\lambda_i)+\sum_{i<j=1}^{N}V_2(\lambda_i,\lambda_j), \tag{2.60}$$

with one- (V_1) and two-body (V_2) interaction terms and summation taken over N degrees of freedom. An example of (2.59) was derived in (2.21) for the dGUE. The degrees of freedom λ are either the eigenvalues or the singular values of underlying dynamical matrices. For equations of type (2.59), one finds an evolution formula for the macroscopic (i.e. after taking $N\to\infty$ limit) eigenvalue density ρ:

$$\begin{aligned}\partial_\tau\rho(x;\tau)&=-\partial_x J(x;\tau),\\ J(x;\tau)&=-\rho(x;\tau)\partial_x\left(\hat{V}_1(x)+\int_I\rho(y;\tau)V_2(x,y)dy\right).\end{aligned} \tag{2.61}$$

where $\hat{V}_1\equiv\lim_{N\to\infty}\frac{1}{N}V_1$ is the one-point potential non-vanishing in the limit and the time variable is rescaled as $Nt=\tau$. The continuity equation (2.61) is valid in the large

matrix size limit (Dyson, 1972) as only then the equation (2.61) is closed as higher order correlation functions decouple. Concrete form of the macroscopic eigenvalue density ρ depends on the underlying dynamical model, we discuss several cases below.

dGβE case

To proceed further, we shall focus on the model of dGβE for $\beta = 1, 2$ where the potential reads $W = -\sum_{j<k} \log|\lambda_j - \lambda_k|$ and so

$$V_1(x) = 0, \quad V_2(x, y) = -\log|x - y|. \tag{2.62}$$

The SFP equation for dGUE ($\beta = 2$) was already given in (2.21). In this setting we define the spectral density as

$$\rho_N(\lambda; \tau) \equiv \frac{1}{N} \left\langle \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \right\rangle, \tag{2.63}$$

along with the Green's function:

$$G_N(z; \tau) \equiv \int_I \frac{\rho_N(y; \tau)}{z - y}\, dy, \tag{2.64}$$

where $I \subseteq \mathbb{R}$ is the support of ρ_N. We skip the N subscript to denote macroscopic functions i.e. $\rho(x; \tau) \equiv \lim_{N\to\infty} \rho_N(x; \tau)$ and $G(z; \tau) \equiv \lim_{N\to\infty} G_N(z; \tau)$. In this limit, the set I becomes bounded. Additionally, we introduce the Hilbert (or Cauchy) transform as the principal value integral:

$$\mathcal{H}[v](x) \equiv \mathrm{PV} \int_I \frac{v(y)}{x - y}\, dy, \qquad x \in I. \tag{2.65}$$

along with an identity:

$$\partial_x \int_I f(x') \log|x - x'| dx' = \mathrm{PV} \int_I \frac{f(x')}{x - x'} dx', \qquad x \in I, \tag{2.66}$$

to which a simple proof is given in Appendix C.1. The hydrodynamical equation (2.61) for the macroscopic density ρ reads

$$\partial_\tau \rho(x; \tau) = -\partial_x \big(\rho(x; \tau) \mathcal{H}[\rho](x; \tau)\big). \tag{2.67}$$

Next we follow the working in (Beenakker, 1997), which begins by noting that as a consequence of the residue theorem, the macroscopic Green's function is related to the Hilbert transform by

$$G_\pm(x; \tau) = \mp i\pi \rho(x; \tau) + \mathcal{H}[\rho](x; \tau), \qquad x \in I, \tag{2.68}$$

where $G_\pm(x; \tau) = \lim_{\epsilon\to 0_+} G(x \pm i\epsilon; \tau)$. Using this relation in (2.67) gives

$$2\partial_\tau (G_-(x; \tau) - G_+(x; \tau)) = -\partial_x ((G_-(x; \tau))^2 - (G_+(x; \tau))^2). \tag{2.69}$$

It must therefore be that the function $2\partial_\tau G(z; \tau) + \partial_z (G(z; \tau))^2$ is analytic throughout the entire complex plane. But according to (2.64), $G(z; \tau) \sim 1/z$ as $|z| \to \infty$, so this function furthermore goes to zero at infinity. The only analytic function with this

property is the zero function, and so after minor manipulation we have

$$\partial_\tau G(z;\tau) + G(z;\tau)\partial_z G(z;\tau) = 0, \tag{2.70}$$

we recover the complex Burgers' equation (2.55) albeit now it is given for $\beta = 1, 2$. It is an instance of (2.56) with $A = 1, B = G$ and $C = 0$, we solve the initial value problem by complex characteristics method. We read off from (2.56) and (2.58) that the differential equations describing characteristic lines and the propagation of the solution are

$$\frac{d}{d\beta} z(\alpha,\beta) = G(\alpha,\beta), \qquad \frac{d}{d\beta}\tau(\alpha,\beta) = 1, \qquad \frac{d}{d\beta} G(\alpha,\beta) = 0. \tag{2.71}$$

The initial data comprises of the initial position $z(\alpha, 0) = \alpha, \tau(\alpha, 0) = 0$ and the starting Green's function $G(\alpha, 0) = G(z(\alpha, 0); \tau(\alpha, 0)) = G(\alpha; 0)$. Explicit integration gives

$$\begin{aligned} G(\alpha,\beta) &= G(\alpha;0), \\ \tau(\alpha,\beta) &= \beta, \\ z(\alpha,\beta) &= \alpha + \beta G(\alpha;0). \end{aligned} \tag{2.72}$$

These, after eliminating α and β, yield the functional equation

$$G(z;\tau) = G\Big(z - \tau G(z;\tau); 0\Big), \tag{2.73}$$

which is sometimes written, recalling the definition (2.64), as an implicit integral equation

$$G(z;\tau) = \int_{I^{(0)}} \frac{\rho^{(0)}(\mu)d\mu}{z - \tau G(z;\tau) - \mu}, \tag{2.74}$$

where $I^{(0)}$ is the support of initial spectral density $\rho^{(0)}$. Working closely related to the above discussion can be found in (Blaizot et al., 2015a). In Fig. 2.1 we present the evolution of eigenvalue density ρ for different initial conditions $\rho^{(0)}(\mu) = \delta(\mu)$ and $\rho^{(0)} = \frac{1}{2}(\delta(\mu-1) + \delta(\mu+1))$. These two scenarios were considered in Sec. 2.3 to calculate universal behaviour of averaged characteristic polynomials in the dGUE case both near the spectral edge (2.34) and in the vicinity of collision (2.37).

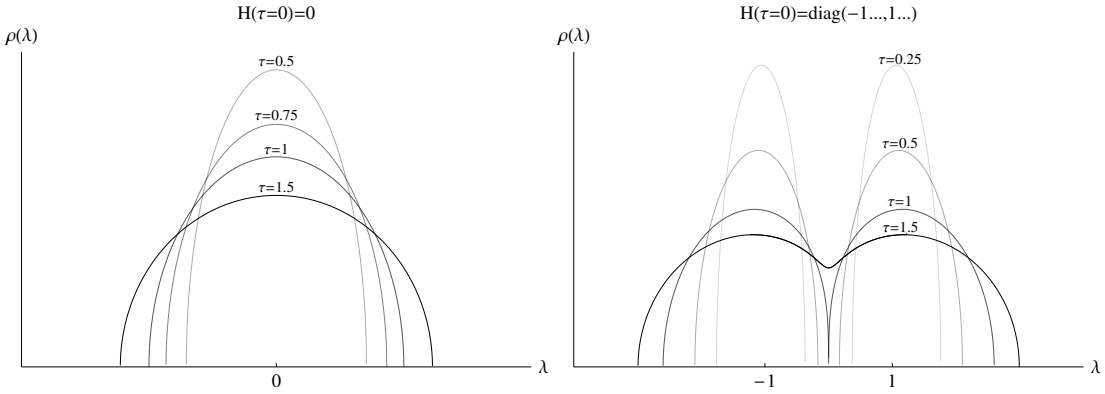

FIGURE 2.1: Time evolution of the macroscopic spectral density ρ of dGβE matrices for two different initial conditions $\rho^{(0)}$. On the left we set $\rho^{(0)}(\mu) = \delta(\mu)$ whereas on the right $\rho^{(0)} = \frac{1}{2}(\delta(\mu-1) + \delta(\mu+1))$.

Other dynamical models

We revisit several other dynamical models to strengthen the robustness of the hydrodynamical picture and present Burgers' equations along with implicit formulae for the solutions.

- **dynamical chiral GβE**
  Object of study is the matrix $M_{m\times n}$ $(n > m)$ without any symmetry constraints. Relevant degrees of freedom are the singular values $x_i \in \mathbb{R}_+$ of M. One- and two-point potentials are given by:

$$V_1(x) = -\frac{a'}{2}\log x^2, \quad V_2(x,y) = -\log|x^2 - y^2|, \tag{2.75}$$

  where $a' = n - m + 1 - \frac{1}{\beta}$. A suitable eigenvalue density in this case reads:

$$\rho_m^{\mathrm{c}}(x;\tau) \equiv \frac{1}{m}\left\langle \sum_{i=1}^{m} (\delta(x - x_i) + \delta(x + x_i)) \right\rangle, \tag{2.76}$$

  along with the corresponding Green's function:

$$G_m^{\mathrm{c}}(z;\tau) \equiv \int_{I_+\cup -I_+} \frac{\rho_m^{\mathrm{c}}(y;\tau)}{z - y}\, dy. \tag{2.77}$$

  where the set $I_+ \subseteq \mathbb{R}_+$ becomes bounded in the large n, m limit. The second Dirac delta function in (2.76) is included to retain the reflection $x \to -x$ symmetry which ease the calculations — in the end we only consider $x > 0$ for which the density is normalized to unity $\int_0^\infty \rho_m^{\mathrm{c}}(x)dx = 1$. Accordingly, the definition of Green's function (2.77) has the same feature. Such choice is also motivated by the chiral context in which each singular value x_i compose a pair of eigenvalues $(x_i, -x_i)$ of the block matrix constructed out of M. The macroscopic density ρ^{c} and G^{c} are defined by taking $m, n \to \infty$ limit.

  To use the continuity formula (2.61), we first calculate

$$\hat{V}_1(x) = \lim_{m\to\infty} \frac{a'}{2m}\log x^2 = \frac{\hat{a}}{2}\log x^2, \quad \hat{a} = r - 1, \tag{2.78}$$

  where $r = \lim_{n,m\to\infty} n/m$ is the rectangularity of the matrix M. The evolution equation for macroscopic density ρ^{c} thus reads

$$\begin{aligned}\partial_\tau \rho^{\mathrm{c}}(x;\tau) &= \partial_x\Big(\rho^{\mathrm{c}}(x;\tau)\partial_x\Big(-\frac{\hat{a}}{2}\log x^2 - \int_{I_+}\rho^{\mathrm{c}}(y;\tau)\log|x^2 - y^2|\,dy\Big)\Big)\\ &= \partial_x\Big(\rho^{\mathrm{c}}(x;\tau)\partial_x\Big(-\frac{\hat{a}}{2}\log x^2 - \int_{I_+\cup -I_+}\rho^{\mathrm{c}}(y;\tau)\log|x - y|\,dy\Big)\Big),\end{aligned} \tag{2.79}$$

  where we utilized the symmetry of both $\rho^{\mathrm{c}}(-x;\tau) = \rho^{\mathrm{c}}(x;\tau)$ and $\log|x^2 - y^2| = \log|x - y| + \log|x + y|$.

  By using the Hilbert transform as defined in (2.65) and the identity (2.66), the formula (2.79) is expressed as

$$\partial_\tau \rho^{\mathrm{c}}(x;\tau) = -\partial x\Big(\rho^{\mathrm{c}}(x;\tau)\mathcal{H}[\hat{a}\delta + \rho^{\mathrm{c}}](x;\tau)\Big). \tag{2.80}$$

where δ is the Dirac delta function. The Green's function and Hilbert transform are related through

$$G^{\mathrm{c}}_{\pm}(x;\tau) = \mp i\pi\rho^{\mathrm{c}}(x;\tau) + \mathcal{H}[\rho^{\mathrm{c}}](x;\tau), \qquad x \in I_+ \cup -I_+. \tag{2.81}$$

Proceeding as in the derivation of (2.70), it follows from the use of (2.81) in (2.80) that

$$\partial_\tau G^{\mathrm{c}}(z;\tau) - \frac{\hat{a}}{z^2} G^{\mathrm{c}}(z;\tau) + \Big(G^{\mathrm{c}}(z;\tau) + \frac{\hat{a}}{z}\Big)\partial_z G^{\mathrm{c}}(z;\tau) = 0. \tag{2.82}$$

We write down an initial value problem of (2.82) as:

$$G^{\mathrm{c}}(z;\tau) = 2\int_{I_+^{(0)}} \frac{\rho^{\mathrm{c},(0)}(\mu)d\mu}{z - G^{\mathrm{c}}(z;\tau)\tau - \frac{2\hat{a}\tau}{z} - \frac{\mu^2}{z - G^{\mathrm{c}}(z;\tau)\tau}}, \tag{2.83}$$

where $\rho^{\mathrm{c},(0)}(\mu)$ is the initial spectral density with support on $I_+^{(0)}$. For derivation we redirect interested reader to (Forrester and Grela, 2016).

- **dynamical chiral GβE with zero modes**
  Now the object of study is the block matrix

$$B_{(n+m)\times(n+m)} = \begin{pmatrix} 0_{m\times m} & M_{m\times n} \\ M^\dagger_{m\times n} & 0_{n\times n} \end{pmatrix}, \tag{2.84}$$

with M as defined before in the dynamical chGβE model. Degrees of freedom are the eigenvalues of B and are related to the singular values of M. The $2m + (n-m)$ eigenvalues of B comprise of m pairs of singular values $(x_i, -x_i)$ and $n-m$ zero eigenvalues (also known as zero modes). Therefore, the one- and two-point potential is also given by (2.75). The relevant spectral density is however modified and reads

$$\rho^{\mathrm{ch}}_m(\lambda;\tau) \equiv \frac{1}{m+n}\left\langle \sum_{i=1}^{m}\Big(\delta(\lambda - x_i) + \delta(\lambda + x_i)\Big) + (n-m)\delta(\lambda)\right\rangle, \tag{2.85}$$

where we pick up all the eigenvalues of B. It is properly normalized to unity $\int_{-\infty}^{\infty}\rho^{\mathrm{ch}}_N(\lambda;\tau)d\lambda = 1$. The Green's function is likewise redefined as

$$g^{\mathrm{ch}}_m(z;\tau) \equiv \int_{I_+\cup -I_+} \frac{\rho^{\mathrm{ch}}_m(x;\tau)}{z-x}\,dx. \tag{2.86}$$

The macroscopic objects are defined analogously as $\rho^{\mathrm{ch}} = \lim_{m\to\infty}\rho^{\mathrm{ch}}_m$ and $g^{\mathrm{ch}} = \lim_{m\to\infty} g^{\mathrm{ch}}_m$. Since we did not really change the degrees of freedom in comparison with the chGβE model, the definitions of (2.76) (2.77), (2.86) and (2.85) are related:

$$\rho^{\mathrm{ch}}(x;\tau) = \frac{1}{2+\hat{a}}\rho^{\mathrm{c}}(x;\tau) + \frac{\hat{a}}{2+\hat{a}}\delta(x), \tag{2.87}$$

$$g^{\mathrm{ch}}(z;\tau) = \frac{1}{2+\hat{a}}G^{\mathrm{c}}(z;\tau) + \frac{\hat{a}}{2+\hat{a}}\frac{1}{z}. \tag{2.88}$$

with $\hat{a}$ defined in (2.78). By plugging the relation (2.88) into (2.82) we obtain the Burgers' equation for g^{ch}:

$$\partial_\tau g^{\mathrm{ch}}(z;\tau) + (2+\hat{a}) g^{\mathrm{ch}}(z;\tau)\partial_z g^{\mathrm{ch}}(z;\tau) + \frac{\hat{a}^2}{(2+\hat{a})}\frac{1}{z^3} = 0, \tag{2.89}$$

which was considered in application to QCD by (Blaizot, Nowak, and Warchoł, 2013a). From our point of view, this equation is completely equivalent to (2.82). For completeness, we write down the initial value solution to (2.89):

$$g^{\mathrm{ch}}(z;\tau) = 2F_{\mathrm{ch}} \int_{I_+^{(0)}} \frac{\rho^{\mathrm{ch},(0)}(\mu)d\mu}{(F_{\mathrm{ch}})^2 - \frac{F_{\mathrm{ch}}\hat{a}\tau}{z} - \mu^2} + \frac{\hat{a}}{2+\hat{a}}\left(\frac{1}{z} - \frac{2z}{zF_{\mathrm{ch}} - \hat{a}\tau}\right), \tag{2.90}$$

where $F_{\mathrm{ch}} = z - \tau(2+\hat{a})g^{\mathrm{ch}}$.

- **dynamical Wishart β Ensemble**
  Now we look at a matrix $W_{m\times m} = M_{m\times n}M^\dagger_{n\times m}$ where M is defined as before. We study the eigenvalues $\lambda_i > 0$ of W which are also related to the chiral models as $\lambda_i = x_i^2$. The potentials are thus also the same, the difference lies in the spectral density which is now given by

$$\rho^{\mathrm{W}}_m(\lambda;\tau) \equiv \frac{1}{m}\left\langle \sum_{i=1}^{m} \delta(\lambda - x_i^2) \right\rangle, \tag{2.91}$$

and normalized properly $\int_0^\infty \rho^{\mathrm{W}}_N(\lambda;\tau)d\lambda = 1$. The corresponding Green's function is defined as:

$$g^{\mathrm{W}}_m(z;\tau) \equiv \int_{I_+} \frac{\rho^{\mathrm{W}}_m(x;\tau)}{z-x}\,dx. \tag{2.92}$$

with $I_+ \subseteq \mathbb{R}_+$. Just as in previous cases, macroscopic objects are denoted by skipping the m subscript. Relation to (2.76) and (2.77) is now the following:

$$\rho^{\mathrm{W}}(\lambda;\tau) = \frac{1}{2\sqrt{\lambda}}\rho^{\mathrm{c}}(\sqrt{\lambda};\tau). \tag{2.93}$$

$$g^{\mathrm{W}}(z^2;\tau) = \frac{G^{\mathrm{c}}(z;\tau)}{2z}. \tag{2.94}$$

Substituting (2.94) into (2.82) gives the Burgers' like equation for g^{W}:

$$\frac{\hat{a}+1}{2}\partial_\tau g^{\mathrm{W}}(z;\tau) + \Big(\hat{a} + 2zg^{\mathrm{W}}(z;\tau)\Big)\partial_z g^{\mathrm{W}}(z;\tau) + [g^{\mathrm{W}}(z;\tau)]^2 = 0. \tag{2.95}$$

This variant, modulo some rescaling, is the one given in (Cabanal Duvillard and Guionnet, 2001; Blaizot, Nowak, and Warchoł, 2013b). Solution to the initial problem of (2.95) is attainable by the method of characteristics:

$$g^{\mathrm{W}}_0(z) = F^{\mathrm{W}} \int_{I_+^{(0)}} \frac{\rho^{\mathrm{W},(0)}(\mu)d\mu}{z(F^{\mathrm{W}})^2 - 2\hat{a}\tau F^{\mathrm{W}} - \mu}, \tag{2.96}$$

with $F^{\mathrm{W}} = 1 - 2\tau g^{\mathrm{W}}(z;\tau)$, the initial eigenvalue support $I_+^{(0)}$ and initial eigenvalue density $\rho^{\mathrm{W},(0)}(\mu)$ is normalized to unity when integrating over $\mu > 0$. This derivation is found in (Forrester and Grela, 2016).

- **dynamical Circular β Ensemble**
  The object of study is a unitary matrix $M_{N\times N}$ with symmetric ($\beta = 1$), unrestricted ($\beta = 2$) or self-dual ($\beta = 4$) constraint. The degrees of freedom are the eigenvalues $e^{i\phi_j}$ of M which lie on the unit circle so that $\phi_i \in (-\pi, \pi]$. The one- and two-point potentials read

$$V_1(\phi) = 0, \quad V_2(\phi, \theta) = -\log|e^{i\phi} - e^{i\theta}|. \tag{2.97}$$

  We formulate the spectral density of phases as

$$\rho_N^\circ(\phi;\tau) \equiv \frac{1}{N}\left\langle \sum_{i=1}^{N} \delta(\phi - \phi_i) \right\rangle, \tag{2.98}$$

  normalized to unity $\int_{-\pi}^{\pi} d\phi \rho_N^\circ(\phi;\tau) = 1$ and introduce a circular Green's function

$$G_N^\circ(z;\tau) \equiv \frac{1}{2}\int_{\bar{I}} \cot\left(\frac{z-y}{2}\right) \rho_N^\circ(y;\tau)\, dy, \tag{2.99}$$

  with the support $\bar{I} \subseteq (-\pi, \pi]$. In the large N limit we define macroscopic functions ρ° and G°. We compute the hydrodynamical continuity equation (2.61) as

$$\partial_\tau \rho^\circ(\phi;\tau) = -\partial_\phi\left[\rho^\circ(\phi;\tau)\partial_\phi\left(\int_{\bar{I}} d\phi' \log|e^{i\phi} - e^{i\phi'}|\rho^\circ(\phi';\tau)\right)\right]. \tag{2.100}$$

  and define a circular Hilbert transform

$$\mathcal{H}_\circ[f](\phi) \equiv \frac{1}{2}\mathrm{PV}\int_I d\phi' \cot\left(\frac{\phi-\phi'}{2}\right) f(\phi'), \qquad \phi \in \bar{I}, \tag{2.101}$$

  where the trigonometric function arose since $\partial_x \log|e^{ix} - e^{iy}| = \frac{1}{2}\cot\left(\frac{x-y}{2}\right)$. With definition (2.101), the equation (2.100) is expressed as

$$\partial_\tau \rho^\circ(\phi;\tau) = -\partial_\phi\left(\rho^\circ(\phi;\tau)\mathcal{H}_\circ[\rho^\circ](\phi;\tau)\right). \tag{2.102}$$

  Since the Green's function of (2.99) and the Hilbert transform (2.101) satisfy the formula (2.68), we again follow the same steps and find the complex Burgers' equation

$$\partial_\tau G^\circ(z;\tau) + G^\circ(z;\tau)\partial_z G^\circ(z;\tau) = 0, \tag{2.103}$$

  which is formally in the same form as the dGβE case (2.70). We can thus apply the same techniques to conclude that the solution of the initial value problem for this equation reads

$$G^\circ(z;\tau) = \int_{\bar{I}^{(0)}} \rho^{\circ,(0)}(\mu)\cot\left(\frac{z - \tau G^\circ(z;\tau) - \mu}{2}\right) d\mu, \tag{2.104}$$

  where $\bar{I}^{(0)}$ is the initial support of $\rho^{\circ,(0)}$. The hydrodynamical equation (2.100) was first derived by Pandey and Shukla (Pandey and Shukla, 1991), using the hierarchy of equations satisfied by the dynamical correlation functions.

  The particular case $\rho^{\circ,(0)}(\mu) = \delta(\mu)$ was studied in the context of two-dimensional Yang-Mills theory by (Durhuus and Olesen, 1981; Blaizot and Nowak, 2008;

Blaizot and Nowak, 2009b; Neuberger, 2008). Even though there is no closed form solution of (2.104), several analytic features can be exhibited, including an collision effect analogous to that depicted on Fig. 2.1: at a critical value of τ_c, two spectral edges collide as the unit circle on which eigenvalues propagate is compact.

- **dynamical Jacobi β Ensemble**
  Consider a unitary (symmetric for $\beta = 1$, unconstrained by $\beta = 2$ or self dual for $\beta = 4$) matrix S of size $(n+m) \times (n+m)$ with $n \geq m$, divide it into 4 blocks

$$S = \begin{pmatrix} r_{n\times n} & t'_{n\times m} \\ t_{m\times n} & r'_{m\times m} \end{pmatrix}, \tag{2.105}$$

and investigate singular values x_i of sub-block t'. Since $x_i \in (0,1)$, we chose the trigonometric parametrization $x_i = \sin\frac{\phi_i}{2}$ and so $\phi_i \in (0,\pi)$. The potential formulae are given by

$$V_1(\phi) = -\frac{a'}{2}\log\sin^2\frac{\phi}{2} - \frac{b'}{2}\log\cos^2\frac{\phi}{2}, \qquad V_2(\phi,\phi') = -\log\left|\sin^2\frac{\phi}{2} - \sin^2\frac{\phi'}{2}\right|,$$

with $a' = n - m + 1 - \frac{1}{\beta}$ and $b' = \frac{1}{\beta}$. The V_2 potential can be rewritten as

$$V_2(\phi,\phi') = -\ln\left|\sin\left(\frac{\phi-\phi'}{2}\right)\right| - \ln\left|\sin\left(\frac{\phi+\phi'}{2}\right)\right|, \tag{2.106}$$

so that the $\phi \to -\phi$ symmetry is evident. The spectral density we are looking into is

$$\rho^{\mathrm{J}}_m(\phi;\tau) \equiv \frac{1}{m}\left\langle \sum_{i=1}^{m} \left(\delta(\phi-\phi_i) + \delta(\phi+\phi_i)\right)\right\rangle, \tag{2.107}$$

where, just as in (2.76), we included an additional term to retain the $\phi \to -\phi$ symmetry. Likewise, the normalization to unity is preserved only when integrated over the proper interval $\int_0^\pi \rho^{\mathrm{J}}_m(\phi;\tau)d\phi = 1$. The corresponding Green's function reads

$$G^{\mathrm{J}}_m(z;\tau) \equiv \frac{1}{2}\int_{\bar{I}_+\cup-\bar{I}_+} \cot\left(\frac{z-y}{2}\right)\rho^{\mathrm{J}}_m(y;\tau)\,dy, \tag{2.108}$$

with $\bar{I}_+ \subset (0,\pi]$. It is also expressible in terms of (2.98) as:

$$G^{\mathrm{J}}_m(z;\tau) = \frac{1}{2}\int_{\bar{I}_+}\left(\cot\left(\frac{z-y}{2}\right) + \cot\left(\frac{z+y}{2}\right)\right)\rho^{\circ}_m(y;\tau)\,dy. \tag{2.109}$$

As before, macroscopic spectral density and Green's function lack m subscript. As an intermediate step to the continuity equation (2.61) we calculate

$$\hat{V}_1(\phi) = \lim_{m\to\infty}\frac{1}{m}\left(-\frac{a'}{2}\log\sin^2\frac{\phi}{2} - \frac{b'}{2}\log\cos^2\frac{\phi}{2}\right) = -\frac{\hat{a}}{2}\log\sin^2\frac{\phi}{2}, \tag{2.110}$$

where $\hat{a} = r - 1$ and $r = \lim_{n,m\to\infty} n/m$. We find the hydrodynamical equation

$$\partial_\tau \rho^{\,\mathrm{J}}(\phi;\tau) = -\partial_\phi \left(\rho^{\,\mathrm{J}}(\phi;\tau) \left(\frac{\hat{a}}{2} \cot\frac{\phi}{2} + \frac{1}{2}\int_{-\pi}^{\pi} d\phi' \cot\frac{\phi-\phi'}{2} \rho^{\,\mathrm{J}}(\phi';\tau) \right) \right) = \\ = -\partial_\phi \Big(\rho^{\,\mathrm{J}}(\phi;\tau) \mathcal{H}_\circ[\hat{a}\delta + \rho^{\,\mathrm{J}}](\phi;\tau) \Big), \tag{2.111}$$

where the Hilbert transform of (2.101) was used. By using the properties of Green's function with $\bar{I}_+ \subset (0,\pi]$, we repeat the derivation of the complex Burgers' equation (2.103) and obtain

$$\partial_\tau G^{\,\mathrm{J}}(z;\tau) + \left(\frac{\hat{a}}{2} \cot\frac{z}{2} + G^{\,\mathrm{J}}(z;\tau) \right) \partial_z G^{\,\mathrm{J}}(z;\tau) - \frac{\hat{a}}{4} \frac{G^{\,\mathrm{J}}(z;\tau)}{\sin^2 z/2} = 0. \tag{2.112}$$

The equation has the same structure as the Burgers' equation (2.82), and in fact reduces to that equation for small z. The underlying Coulomb gas set-up has therefore the same features — it consists of a fixed particle at $\phi = 0$ of charge $\hat{a}$ and two mirror-like clouds for $\phi \in (-\pi, 0)$ and $\phi \in (0, \pi)$ respectively. In the special case of vanishing charge $\hat{a} = 0$, the resulting equation coincides exactly with (2.103) obtained for the dynamical CβE.

2.5.3 Hydrodynamics out of collective variables

Collective variables form another possible route of obtaining the hydrodynamic equations (2.61). The idea of collective variables was first introduced in plasma physics (Bohm and Pines, 1953) and extensively applied to gauge theories (Jevicki and Sakita, 1980) and quantum Hall effect (Laskin, Can, and Wiegmann, 2015). Besides re-deriving the aforementioned hydrodynamical equations, this method is suitable for obtaining asymptotic formulae for group integrals of Harish-Chandra / Itzykson–Zuber and Berezin–Karpelevich type. The former relate to the dGβE case whereas the latter appear in the chGβE.

To proceed, we transform the Smoluchowski–Fokker–Planck equation (2.59) to new collective-type variables $\hat{\lambda}$,

$$\lambda_i \to \hat{\lambda}_j(\{\lambda\}), \qquad \{\lambda\} \equiv (\lambda_1, ..., \lambda_N), \tag{2.113}$$

where $i = 1...N, j = 1...N'$. These new degrees of freedom should a) use the symmetries of the system and b) have a well defined large N limit. For the special case $N' = N$ and N finite, the method is a *bona-fide* variable change. We take however $N' \to \infty$ from the beginning so that the change is not bijective, at least before taking the large N limit.

Consider a general transformation of this type — $\hat{\lambda}(q;\{\lambda\})$ with i index promoted to a variable q (i.e. formally in the limit $N' \to \infty$). To proceed, we express the SFP equation (2.59) for a rescaled function Π_t:

$$P_t\left(\{\lambda\}|\{\lambda^0\}\right) = e^{-\beta W(\{\lambda\})} \Pi_t\left(\{\lambda\}|\{\lambda^0\}\right), \tag{2.114}$$

with W defined in (2.60). Following the redefinition, an SFP equation for Π_t reads

$$\partial_t \Pi_t\left(\{\lambda\}|\{\lambda^0\}\right) = L\Pi_t\left(\{\lambda\}|\{\lambda^0\}\right), \qquad L \equiv K + V, \tag{2.115}$$

with operators K and V given by

$$K \equiv \frac{1}{\beta}\sum_{i=1}^{N}\frac{\partial^2}{\partial {\lambda_i}^2}, \qquad V \equiv -\sum_{i=1}^{N}\frac{\partial W}{\partial \lambda_i}\frac{\partial}{\partial \lambda_i}. \tag{2.116}$$

The goal now is to change the variables $\lambda \to \hat{\lambda}$ in the $N' \to \infty$ limit. First of all, the classic chain rule of differential operator becomes a functional derivative in that limit:

$$\frac{\partial}{\partial \lambda_j} = \sum_{i=1}^{N'}\frac{\partial \hat{\lambda}_i(\{\lambda\})}{\partial \lambda_j}\frac{\partial}{\partial \hat{\lambda}_i} \qquad \overset{\text{“}N'\to\infty\text{”}}{\longrightarrow} \qquad \frac{\partial}{\partial \lambda_j} = \int dq \frac{\partial \hat{\lambda}(q;\{\lambda\})}{\partial \lambda_j}\frac{\delta}{\delta \hat{\lambda}(q)}, \tag{2.117}$$

where the $\hat{\lambda}(q)$ are the new variables enumerated by a continuous index q. Consequently, the functions of $\{\lambda\}$ become functionals in the $\hat{\lambda}(q)$ variables:

$$\Pi_t\left(\{\lambda\}|\{\lambda^0\}\right) = \hat{\Pi}_t\left[\hat{\lambda}|\hat{\lambda}^0\right]. \tag{2.118}$$

The transformed operator $\hat{L} = \hat{K} + \hat{V}$ reads

$$\begin{aligned}\hat{K} &= \frac{1}{\beta}\int dq \sum_{i=1}^{N}\frac{\partial^2 \hat{\lambda}(q)}{\partial \lambda_i^2}\frac{\delta}{\delta \hat{\lambda}(q)} + \frac{1}{\beta}\int dpdq \sum_{i=1}^{N}\frac{\partial \hat{\lambda}(q)}{\partial \lambda_i}\frac{\partial \hat{\lambda}(p)}{\partial \lambda_i}\frac{\delta^2}{\delta \hat{\lambda}(p)\delta \hat{\lambda}(q)},\\ \hat{V} &= -\int dq \left(\sum_{i=1}^{N}\frac{\partial W}{\partial \lambda_i}\frac{\partial \hat{\lambda}(q)}{\partial \lambda_i}\right)\frac{\delta}{\delta \hat{\lambda}(q)},\end{aligned} \tag{2.119}$$

where we suppressed the $\{\lambda\}$ dependence in the coefficients. The transformed SFP equation of (2.115) is equal to

$$\partial_t \hat{\Pi}_t\left[\hat{\lambda}|\hat{\lambda}^0\right] = \left(\hat{K} + \hat{V}\right)\hat{\Pi}_t\left[\hat{\lambda}|\hat{\lambda}^0\right], \tag{2.120}$$

which is now a multi-dimensional *functional* PDE for $\hat{\Pi}_t$. In the following sections we will investigate this formula in the large N limit and for particular choice of collective variables $\hat{\lambda}$.

dynamical GβE revisited

In the canonical case of dynamical GβE, we read off the drift term W from (2.60) and (2.62). The collective variable reads

$$\hat{\lambda}(q;\{\lambda\}) \equiv \sum_{i=1}^{N}\delta(q-\lambda_i), \tag{2.121}$$

which is exactly the (non-averaged and up to a constant) eigenvalue density (2.63). Importantly, the variable (2.121) conserves the eigenvalue exchange symmetry. To proceed, we set an ansatz for the functional $\hat{\Pi}_t$

$$\hat{\Pi}_t\left[\hat{\lambda}|\hat{\lambda}^0\right] = \exp\left(-\frac{\beta}{2}N^2 S_t\left[\hat{\lambda}|\hat{\lambda}^0\right]\right). \tag{2.122}$$

As $\hat{\Pi}_t$ solves the SFP equation (2.120), it is a joint probability density function and so $N^2 S_t$ is the Gibbs–Boltzmann weight function of the underlying gas of particles.

Accordingly, S_t satisfies an evolution equation

$$\partial_t S_t = \int dp\, \hat{\lambda}(p) \left(\frac{1}{\beta} \frac{\partial^2}{\partial p^2} \left(\frac{\delta S_t}{\delta \hat{\lambda}(p)} + \frac{\delta^2 S_t}{\delta \hat{\lambda}(p)^2} \right) - \mathcal{H}[\delta](0) \frac{\partial}{\partial p} \frac{\delta S_t}{\delta \hat{\lambda}(p)} \right) + \\ - \int dp\, \hat{\lambda}(p) \left(\frac{N^2}{2} \left(\frac{\partial}{\partial p} \frac{\delta S_t}{\delta \hat{\lambda}(p)} \right)^2 - \mathcal{H}[\hat{\lambda}](p) \frac{\partial}{\partial p} \frac{\delta S_t}{\delta \hat{\lambda}(p)} \right), \tag{2.123}$$

where $\mathcal{H}[f]$ denotes the Hilbert transform (2.65) with suppressed arguments. This result is derived in the Appendix C.2.

Now we are ready to perform the large N limit of (2.123) by rescaling both the time $Nt = \tau$ and the collective variable $\hat{\lambda} = N\rho$. At this stage large N collective variables become macroscopic densities and S_t converges to the systems' free energy. On a technical level, we conduct large N limit by skipping the first term on the r.h.s. of (2.123) as sub-leading in comparison to the remaining ones and find that:

$$\partial_\tau S_\tau + \int dp\, \rho(p) \left[\frac{1}{2} \left(\frac{\partial}{\partial p} \frac{\delta S_\tau}{\delta \rho(p)} \right)^2 - \mathcal{H}[\rho](p) \frac{\partial}{\partial p} \frac{\delta S_\tau}{\delta \rho(p)} \right] = 0. \tag{2.124}$$

This equation is in the Hamilton–Jacobi form — the position variable reads $\rho(p)$ whereas the conjugate momentum is given as $\Delta(p) = \frac{\delta S_\tau}{\delta \rho(p)}$. We interpret S_τ as an action evaluated on a physical trajectory between two macroscopic densities $\rho(p; \tau = 0)$ and $\rho(p; \tau)$. The resulting Hamiltonian is of hydrodynamical nature:

$$H = \int dp\, \rho \left(\frac{1}{2} (\partial_p \Delta)^2 - \mathcal{H}[\rho] \partial_p \Delta \right), \tag{2.125}$$

where we suppressed the arguments of $\rho = \rho(p; \tau)$ and $\Delta = \Delta(p; \tau)$ and will continue to do so from now on. To cancel the problematic Hilbert transform term we invoke a canonical change of variables $(\rho, \Delta) \to (\rho', \Delta' = \Delta + C)$ with C dependent only on ρ. The details are given out in (Forrester and Grela, 2016) where we present only the transformed Hamiltonian:

$$H[\rho, \Delta] = \frac{1}{2} \int dp\, \rho \left((\partial_p \Delta)^2 - \frac{\pi^2}{3} \rho^2 \right), \tag{2.126}$$

with the corresponding action

$$S_{\tau'} = \frac{1}{2} \int_0^{\tau'} d\tau \int dp\, \rho \left((\partial_p \Delta)^2 + \frac{\pi^2}{3} \rho^2 \right), \tag{2.127}$$

chosen so that $S_\tau|_{\tau=0} = 0$ (otherwise S_τ is unique only up to an additive constant). By the Hamilton equations $\partial_\tau \rho = \frac{\delta H}{\delta \Delta}, \partial_\tau \Delta = -\frac{\delta H}{\delta \rho}$, the equations of motion read

$$\partial_{\hat{\tau}} \Delta + \frac{1}{2} (\partial_p \Delta)^2 = \frac{\pi^2}{2} \rho^2, \\ \partial_\tau \rho + \partial_p (\rho \partial_p \Delta) = 0, \tag{2.128}$$

which define a $(1+1)$ hydrodynamical system for the macroscopic density $\rho(p; \tau)$ and its canonical momentum $\Delta(p; \tau)$. Firstly, upon defining $G_\pm = \mp i\pi\rho + \partial_p \Delta$ (compare with (2.68)), equation (2.128) reproduces exactly the complex Burgers' equation (2.70). The system of (2.128) is however richer — in approaches described in Chapters 2.5.1

and 2.5.2, the Green's function $G_\pm$ depends only on the spectral density ρ (compare with (2.68)). This results in solutions (2.74) to Burgers' equation dependent only on the initial eigenvalue density $\rho^{(0)}$. In this framework however, a full fluid-like arise — $G_\pm$ depends on both the density ρ and its canonical momentum Δ. Therefore, a model arise in which both the initial spectral density $\rho^{(0)}$ and the initial canonical momentum $\Delta^{(0)}$ determine the evolution.

These results are well-known from the work of Matytsin (Matytsin, 1994), reproduced also by other authors (Guionnet, 2003; Bun et al., 2014). Here we show how additionally the joint PDF function $\hat{\Pi}$ is asymptotically expressed in terms of an action related to the hydrodynamical system.

dynamical chGβE revisited

For the chiral case we replace $\lambda \to x$ and read off the potential W from the equations (2.60) and (2.75). The collective variable in this case is

$$\hat{x}(q,\{x\}) \equiv \sum_{i=1}^{m} (\delta(q-x_i)+\delta(q+x_i)) = \sum_{i=1}^{m} 2|q|\delta(q^2-x_i^2), \tag{2.129}$$

which is the non-averaged eigenvalue density of (2.76). We utilize the functional SFP equation of (2.120) with replacement $\hat{\Pi}_t \to \hat{\Pi}_t^{\mathrm{c}}$ and make an ansatz

$$\hat{\Pi}_t^c\left[\hat{x}|\hat{x}^0\right] = \exp\left(-\frac{\beta}{4}m^2 S_t^{\mathrm{c}}\left[\hat{x}|\hat{x}^0\right]\right). \tag{2.130}$$

The equation satisfied by S_t^c reads

$$\begin{aligned}\partial_t S_t^{\mathrm{c}} = &\int dp\,\hat{x}(p)\left(\frac{1}{\beta}\frac{\partial^2}{\partial p^2}\left(\frac{\delta S_t^{\mathrm{c}}}{\delta\hat{x}(p)}+2\frac{\delta^2 S_t^{\mathrm{c}}}{\delta\hat{x}(q)^2}\right)+\mathcal{H}[\delta](0)\frac{\partial}{\partial p}\frac{\delta S_t^{\mathrm{c}}}{\delta\hat{x}(p)}\right)+\\ &-\int dp\,\hat{x}(p)\left(\frac{m^2}{2}\left(\frac{\partial}{\partial p}\frac{\delta S_t^{\mathrm{c}}}{\delta\hat{x}(p)}\right)^2-\mathcal{H}[a'\delta+\hat{x}](p)\frac{\partial}{\partial p}\frac{\delta S_t^{\mathrm{c}}}{\delta\hat{x}(p)}\right),\end{aligned} \tag{2.131}$$

with $\mathcal{H}$ Hilbert transform defined by (2.65) and the details of the derivation are given in Appendix C.3.

Now we perform the $m,n\to\infty$ limit with $n/m=r$ fixed. We set $\hat{x}=m\rho^{\mathrm{c}}, \tau = mt$ and find the first term on r.h.s. sub-leading with respect to the remaining ones. The equation for S_τ^{c} is again in the Hamilton–Jacobi form

$$\partial_\tau S_\tau^{\mathrm{c}} + \int dp\,\rho^{\mathrm{c}}(p)\left(\frac{1}{2}\left(\frac{\partial}{\partial p}\frac{\delta S_\tau^{\mathrm{c}}}{\delta\rho^{\mathrm{c}}(p)}\right)^2-\frac{\hat{a}}{p}\frac{\partial}{\partial p}\frac{\delta S_\tau^{\mathrm{c}}}{\delta\rho^{\mathrm{c}}(p)}-\mathcal{H}[\rho^{\mathrm{c}}](p)\frac{\partial}{\partial p}\frac{\delta S_\tau^{\mathrm{c}}}{\delta\rho^{\mathrm{c}}(p)}\right)=0. \tag{2.132}$$

with the conjugate momentum $\Delta^{\mathrm{c}} = \frac{\delta S_\tau^{\mathrm{c}}}{\delta\rho^{\mathrm{c}}}$. We again make a canonical transformation and obtain the Hamiltonian

$$H = \frac{1}{2}\int dp\,\rho^{\mathrm{c}}\left((\partial_p\Delta^{\mathrm{c}})^2-\frac{\pi^2}{3}(\rho^{\mathrm{c}})^2+\frac{2\hat{a}}{p}\partial_p\Delta^{\mathrm{c}}\right), \tag{2.133}$$

where $\hat{a} = r - 1$ and the arguments were suppressed $\rho^{\rm c} = \rho^{\rm c}(p;\tau)$ and $\Delta^{\rm c} = \Delta^{\rm c}(p;\tau)$. The corresponding action reads

$$S^{\rm c}_{\tau'} = \frac{1}{2}\int_0^{\tau'} d\tau \int dp\, \rho^{\rm c}\left((\partial_p\Delta^{\rm c})^2 + \frac{\pi^2}{3}(\rho^{\rm c})^2 + \frac{2\hat{a}}{p}\partial_p\Delta^{\rm c}\right). \tag{2.134}$$

The details of this calculation are given in the source paper (Forrester and Grela, 2016). The corresponding Hamilton equations of motion read

$$\begin{aligned}&\partial_\tau\Delta^{\rm c} + \frac{1}{2}(\partial_p\Delta^{\rm c})^2 + \frac{\hat{a}}{p}\partial_p\Delta^{\rm c} = \frac{\pi^2}{2}(\rho^{\rm c})^2,\\ &\partial_\tau\rho^{\rm c} + \partial_p(\rho^{\rm c}\partial_p\Delta^{\rm c}) + \partial_p\left(\frac{\hat{a}}{p}\rho^{\rm c}\right) = 0.\end{aligned} \tag{2.135}$$

We observe again how the chiral case reduces to the dGβE (2.128) when $\hat{a} = 0$, and by defining $G^{\rm c}_{\pm} = \mp i\pi\rho^{\rm c} + \partial_p\Delta^{\rm c}$ we reclaim (2.82) by the arguments elucidated previously.

Asymptotic expansion of HC/IZ and BK integrals

Collective variables were used by Matytsin (Matytsin, 1994) to obtain large N expansion of the celebrated Harish-Chandra/Itzykson–Zuber integral formula. In this Chapter we comment on this standard result and afterwards report on an expansion of the Berezin–Karpelevich type integrals (Berezin and Karpelevich, 1958; Guhr and Wettig, 1996) arising in the chiral ensembles.

HC/IZ-type integrals. We consider an integral

$$I_\beta(A,B) \equiv \int (U^\dagger dU)\exp\left(\frac{\beta N}{2}\mathrm{Tr}(UAU^\dagger B)\right), \tag{2.136}$$

where matrices A, B are diagonal of size $N\times N$ and the matrix U is either real orthogonal ($\beta = 1$) or complex unitary ($\beta = 2$). In the random matrix theory context these integrals arise in connection with the Gaussian ensembles. For $\beta = 2$ an exact formula exists, found independently by Harish-Chandra (Harish-Chandra, 1957) and Itzykson–Zuber (Itzykson and Zuber, 1980).

To obtain the large N asymptotic behaviour of (2.136), we first recall the definition of Π_τ (2.114) and the jPDF (2.17) which combine into

$$\Pi_\tau\left(L|L^{(0)}\right) = \int (U^\dagger dU) P_\tau\left(ULU^\dagger|L^{(0)}\right) = \frac{1}{C^\beta_{N,\tau}} e^{-\frac{\beta N}{4\tau}\mathrm{Tr}L^2 - \frac{\beta N}{4\tau}\mathrm{Tr}(L^{(0)})^2} I_\beta\left(\frac{L}{\sqrt{\tau}}, \frac{L^{(0)}}{\sqrt{\tau}}\right), \tag{2.137}$$

where $L, L^{(0)}$ are the diagonal matrices of eigenvalues.

The traces in this formula are re-expressed in terms of the collective variable (2.121) as $\mathrm{Tr}L^2 = N\int dp\, p^2\rho(p;\tau)$ and $\mathrm{Tr}(L^{(0)})^2 = N\int dp\, p^2\rho(p;0)$ in accordance to their role as a final and initial densities respectively. On the other hand, the asymptotic form of Π_τ is known to behave as

$$\Pi_\tau \sim \exp\left(-\frac{\beta}{2}N^2\left(S_\tau - T_{|\tau} - T_{|0}\right)\right), \tag{2.138}$$

with $T = -\frac{1}{2}\int dpdq\rho(p;\tau)\rho(q;\tau)\log|p-q|$. This formula was reported in Chapter 2.5.3 however for the details we relegate to the paper (Forrester and Grela, 2016).

To arrive at an asymptotic expression for (2.136), we fix the time $\tau = 1$ and rename the final $\rho(p;\tau=1) \equiv \rho_f(p)$ and initial $\rho(p;\tau=0) \equiv \rho_i(p)$ densities

$$I_\beta(\sigma,\alpha) \sim C^\beta_{N,\tau=1}\exp\left(\frac{\beta}{2}N^2\left[-S_{\tau=1} + \frac{1}{2}\int dp\, p^2(\rho_i(p)+\rho_f(p)) + \right.\right. \\ \left.\left. -\frac{1}{2}\int dpdq\Big(\rho_i(p)\rho_i(q)+\rho_f(p)\rho_f(q)\Big)\ln|q-p|\right]\right). \tag{2.139}$$

Now the main difficulty lies in finding a physical path joining initial $\rho_i(p)$ and final $\rho_f(p)$ spectral densities and calculating the corresponding action S_τ specified by (2.127).

Berezin–Karpelevich type integrals We now turn to the asymptotic formula for an integral of Berezin–Karpelevich type defined as

$$J_\beta(A,B) \equiv \int (U^\dagger dU)(V^\dagger dV)\exp\left(\frac{\beta m}{4}\mathrm{Tr}\left(VA^\dagger U^\dagger B + B^\dagger UAV^\dagger\right)\right), \tag{2.140}$$

where A, B are $n\times m$ diagonal matrices and the U, V are either real orthogonal ($\beta = 1$) or complex unitary ($\beta = 2$) matrices of sizes $n\times n$ and $m\times m$ respectively with $n > m$.

In the $\beta = 2$ case, an exact formula was calculated by Berezin and Karpelevich (Berezin and Karpelevich, 1958) and rediscovered in (Guhr and Wettig, 1996). To obtain an asymptotic expression for $\beta = 1, 2$, we write down the jPDF for the dynamical chiral GβE:

$$P_\tau(M|M_0) = \frac{1}{C^{\beta,c}_{m,\tau}}\exp\Big(-\frac{m\beta}{4\tau}\mathrm{Tr}(M-M_0)^\dagger(M-M_0)\Big), \tag{2.141}$$

where matrix M is of size $m\times n$ with $n > m$ (see also Chapter 2.5.2). We recall the ansatz (2.130) and obtain

$$\Pi^{\mathrm{c}}_\tau = \int (U^\dagger dU)(V^\dagger dV)P_\tau(UXV^\dagger|X_0) = \\ = \frac{1}{C^{\beta,c}_{m,\tau}}e^{-\frac{\beta m}{4\tau}\mathrm{Tr}\left(X^\dagger X + (X_0)^\dagger X_0\right)}J_\beta\left(\frac{X}{\sqrt{\tau}},\frac{X_0}{\sqrt{\tau}}\right). \tag{2.142}$$

where X and X_0 are the diagonal matrices filled with singular values of M and M_0. respectively. We introduce the collective variable to the Gaussian terms $\mathrm{Tr}X^\dagger X = \frac{m}{2}\int dq\, q^2\rho^c(q;\tau)$ and $\mathrm{Tr}(X_0)^\dagger X_0 = \frac{m}{2}\int dq\, q^2\rho^c(q;\tau=0)$. We write down the asymptotic form of l.h.s. discussed in Chapter 2.5.3:

$$\Pi^{\mathrm{c}}_\tau \sim \exp\left(-\frac{\beta m^2}{4}\left(S^{\mathrm{c}}_\tau - T^{\mathrm{c}}_{|\tau} - T^{\mathrm{c}}_{|0}\right)\right), \tag{2.143}$$

where $T^{\mathrm{c}} = -\frac{1}{2}\int dpdq\rho^{\mathrm{c}}(p;\tau)\rho^{\mathrm{c}}(q;\tau)\log|p-q|$. By comparing (2.142) and (2.143), for a fixed time $\tau = 1$ we obtain an asymptotic formula

$$J_\beta\left(\rho_f^{\mathrm{c}},\rho_i^{\mathrm{c}}\right) \sim C_{m,\tau=1}^{\beta,c}\exp\left(\frac{\beta}{4}m^2\left[-S_{\tau=1}^{\mathrm{c}} + \frac{1}{2}\int dp\, p^2\left(\rho_i^{\mathrm{c}}(p)+\rho_f^{\mathrm{c}}(p)\right) + \right.\right.$$
$$\left.\left. -\frac{1}{2}\int dpdq\left(\rho_i^{\mathrm{c}}(p)\rho_i^{\mathrm{c}}(q)+\rho_f^{\mathrm{c}}(p)\rho_f^{\mathrm{c}}(q)\right)\ln|q-p|\right]\right). \tag{2.144}$$

where the initial and final densities are denoted as $\rho^{\mathrm{c}}(p;\tau=0)\equiv\rho_i^{\mathrm{c}}(p)$ and $\rho^{\mathrm{c}}(p;\tau=1)\equiv\rho_f^{\mathrm{c}}(p)$ respectively. As in the case of (2.139), to obtain the asymptotic formula for prescribed initial ρ_i^{c} and final ρ_f^{c} densities, it is necessary to evaluate the action S_τ^{c} of (2.134) on a physical trajectory connecting these two spectral densities.

Chapter 3

Non-Hermitian dynamical random matrices

So far in this dissertation we have considered dynamical matrices with one common feature — their eigenvalues occupy one-dimensional manifolds. Concrete examples are the real line for the GβE, half-line for chGβE or unit circle in CβE, all of which are dictated by the presence of symmetries. Despite the success of such models, we ask a natural question — what is the behaviour of random matrices without any symmetry constraints? Such inquiry was pursued by Ginibre in his seminal work (Ginibre, 1965). Akin to the classical trio of Gaussian Ensembles, he considered matrices of real, complex and quaternionic numbers — resulting objects are respectively non-symmetric, non-Hermitian and non-self-dual. Perhaps the most characteristic feature is that in these cases the eigenvalues spread over the whole complex plane. Upon closer inspection of the non-Hermitian case in dynamical setting, we have gained a unique perspective and revealed features of non-Hermitian matrix models largely ignored so far. Perhaps the most crucial is the deciding role of eigenvector variables in the dynamical description. This is a surprise as eigenvectors studied so far (*e.g.* (2.25) in the case of dGUE) were largely decoupled from the eigenvalues or at least the relationship was just one-way. In the non-Hermitian world however the picture is radically different — eigenvalues and eigenvectors in dynamical setting are intertwined and can only evolve together. This insight was made largely available due to large matrix size analysis where even though the dynamics simplify considerably, the eigenvectors remain crucial. We looked also into numerical simulations and simple 2×2 problems to gain insight into the dynamics in question.

β	Δ_X operator	Matrix form
1	$\frac{1}{4}\sum_{i,j=1}^{N}\partial^2_{x_{ij}}$	$X_{kl}=x_{kl}$
2	$\frac{1}{4}\sum_{d=1}^{2}\sum_{i,j=1}^{N}\partial^2_{x^{(d)}_{ij}}$	$X_{kl}=x^{(1)}_{kl}+ix^{(2)}_{kl}$
4	$\frac{1}{4}\sum_{d=1}^{4}\sum_{i,j=1}^{N}\partial^2_{x^{(d)}_{ij}}$	$X_{kl}=\sum_{d=1}^{4}x^{(d)}_{kl}\tilde{\sigma}_d$

TABLE 3.1: Spatial operators Δ_X of the SFP equation (3.1) for the dynamical Ginibre β Ensemble (or dGinβE). Real and complex matrices are N dimensional, symplectic case is of size $2N$. The $\tilde{\sigma}_d$'s form a four-vector of 2×2 matrices $(1_2, i\sigma_3, i\sigma_2, i\sigma_1)$ where σ_i are Pauli matrices and $|x|^2_Q$ is the quaternionic norm.

3.1 Dynamical GinβE models

In this chapter we set the dynamical matrix as $M \equiv X$. The evolving of $X_{N\times N}$ without any symmetry constraints (the dynamical Ginibre β Ensemble) admits a representation in terms of an entry-wise SFP equation (2.16):

$$\partial_t P_t\left(X|X_0\right) = \Delta_X P_t\left(X|X_0\right), \tag{3.1}$$

where X_0 is the initial matrix. We assemble the operators Δ_X of dGinβE in Tab. 3.1. As the SFP equation is linear and factorizable, the solution to (3.1) for delta-like initial condition $P^{\beta}_0\left(X|X_0\right) = \delta\left(X - X_0\right)$ is known:

$$P^{\beta}_t\left(X|X_0\right) = \frac{1}{D^{\beta}_{N,t}} \exp\left(-\frac{1}{t}\mathrm{Tr}\,|X - X_0|^2\right), \tag{3.2}$$

where the meaning of the matrix norm $|M|^2$ varies between models - $|M|^2 \to M^T M$ for $\beta = 1$, $|M|^2 \to M^\dagger M$ for $\beta = 2$ and $|M|^2 \to M^D M$ for $\beta = 4$ (where D denotes the quaternionic dual). The constant $D^{\beta}_{N,t}$ is a proper normalization constant. The approach of Dyson discussed in Chapter 2.2.1 is again utilized. Now however, full picture is not available — although we constructed a dynamical model reproducing the entry-wise jPDF of (3.2), evolution equations for the eigenvalues and eigenvectors (analogous to (2.22) and (2.25) in the dGUE case) are not known. Despite our ignorance, based on the entry-wise solution (3.2) we are confident that the dynamical matrices of Tab. 3.1 describe the evolution of Ginibre-like matrices (in the simplest $X_0 = 0$ case, the jPDF of (3.2) is exactly the static model of Ginibre with t dependent variance). The difficulty in obtaining equations for the dynamics of eigenvalues and eigenvectors is perhaps the main reason why the latter were largely dismissed so far. In this Chapter we circumvent this problem by studying certain observables containing eigenvectors in the large N limit.

From now on we focus on the $\beta = 2$ case or the dynamical GinUE model as the simplest yet non-trivial. Just as in the Gaussian case, the initial condition of dynamical model is transcribed into an external source in static random matrices by interpreting (3.2) as a jPDF of the matrix ensemble. We will use this relationship in Chapter 3.5 to arrive at the spectral densities valid for any N and for arbitrary X_0. The lack of dynamical equations for the eigenvalues and eigenvectors translates into few exact results in the static models described by the jPDF of (3.2) for — in particular, for $X_0 \neq 0$ no determinantal (or any other) structure is known to exist, in stark difference to the

dGUE model where the kernel formula of (2.47) in the dynamical regime was calculated.

3.1.1 Quaternionic method and electrostatic analogy

Before continuing with the dynamics, we revise the quaternionic method (Janik et al., 1997; Jarosz and Nowak, 2006) — a powerful tool of obtaining macroscopic results in the non-Hermitian regime. We do not discuss a similar approach of Hermitization devised independently by Feinberg and Zee in (Feinberg and Zee, 1997b; Feinberg and Zee, 1997a). This exposition serves as a motivation and a starting point to what follows.

A backbone of the method lies in the electrostatic analogy (Sommers et al., 1988; Fyodorov and Sommers, 1997; Brown, 1983). We introduce a quantity called the electrostatic potential

$$\Phi_N(z,w) \equiv \frac{1}{N}\left\langle \mathrm{Tr}\log\left(|z-X|^2+|w|^2\right)\right\rangle, \tag{3.3}$$

where the matrix $|z-X|^2 \equiv (z-X)(\bar{z}-X^\dagger)$ and for now the average is taken over static non-Hermitian matrix model of choice. The function Φ_N can be interpreted in the limit $w \to 0$ as an electrostatic log-potential of a system of N electric charges interacting on the z-complex plane. The corresponding Poisson equation reads

$$\lim_{w\to 0}\frac{1}{\pi}\partial_{\bar{z}z}\Phi_N(z,w) = \rho_N(z), \tag{3.4}$$

where on the r.h.s. we introduce charge density producing the potential Φ:

$$\rho_N(z) \equiv \frac{1}{N}\left\langle \sum_i \delta^{(2)}(z-z_i)\right\rangle. \tag{3.5}$$

where z_i's are the complex eigenvalues of X and the complex Dirac delta function is defined as $\delta^{(2)}(x) \equiv \delta(\mathrm{Re}x)\delta(\mathrm{Im}x)$. To arrive at (3.4), we use the representation $\pi\delta^{(2)}(x) = \lim_{w\to 0}\frac{|w|^2}{\left(|w|^2+|x|^2\right)^2}$. The electrostatic analogy serves as a non-Hermitian counterpart of the Green's function formalism summarised below:

	Hermitian	non-Hermitian
potential	$\langle \mathrm{Tr}\log(z-H)\rangle$	$\left\langle \mathrm{Tr}\log\left(\lvert z-X\rvert^2+\lvert w\rvert^2\right)\right\rangle$
	$\Big\downarrow \partial_z$	$\Big\downarrow \partial_z$
Green's function	$\left\langle \mathrm{Tr}\frac{1}{z-H}\right\rangle$	$\left\langle \mathrm{Tr}\frac{\bar{z}-X^\dagger}{\lvert z-X\rvert^2+\lvert w\rvert^2}\right\rangle$
	$\Big\downarrow -\pi^{-1}\lim_{\mathrm{Im}z\to 0^+}\mathrm{Im}$	$\Big\downarrow \pi^{-1}\lim_{w\to 0}\partial_{\bar{z}}$
eigenvalue density	$\left\langle\sum_i \delta(\lambda-\lambda_i)\right\rangle$	$\left\langle\sum_i \delta^{(2)}(z-z_i)\right\rangle$

The quaternions are introduced as a way to cast the non-Hermitian formulae into a form resembling the Hermitian framework. To this end we introduce $2N \times 2N$ matrices

$$Q \equiv \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix}, \quad \mathcal{X} \equiv \begin{pmatrix} X & 0 \\ 0 & X^\dagger \end{pmatrix}, \tag{3.6}$$

where notation is such that the numbers are always implicitly multiplied by an $N \times N$ unit matrix i.e. $z = z\mathbf{1}_N$. We use a quaternionic representation in terms of Pauli matrices σ_i. To this end we form a vector $\tilde{\sigma} = (1_2, i\sigma_3, i\sigma_2, i\sigma_1)$ so that a quaternion is given by $Q = \sum_{i=1}^4 \tilde{\sigma}_i q_i$ with $z = q_1 + iq_2$ and $-\bar{w} = q_3 + iq_4$. As was pointed out in (Burda and Swiech, 2015), the numbers $(z, -\bar{w})$ form an algebraic structure of the Cayley–Dickson pair. Using the quaternions Q, the electrostatic potential of (3.3) reads

$$\Phi_N(z, w) \equiv \Phi_N(Q) = \frac{1}{N} \langle \mathrm{Tr} \log(Q - \mathcal{X}) \rangle, \tag{3.7}$$

and the quaternionic Green's function (hereafter the q-Green's function) is now a 2×2 matrix:

$$\mathcal{G}_N(Q) \equiv \frac{1}{N} \left\langle \mathrm{bTr} \frac{1}{\mathrm{Q} - \mathcal{X}} \right\rangle = \begin{pmatrix} H_N & V_N \\ -\overline{V_N} & \overline{H_N} \end{pmatrix} \tag{3.8}$$

with two independent entries H_N, V_N and the block-trace is defined as

$$\mathrm{bTr} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv \begin{pmatrix} \mathrm{Tr}A & \mathrm{Tr}B \\ \mathrm{Tr}C & \mathrm{Tr}D \end{pmatrix}. \tag{3.9}$$

Using this definition we calculate H_N and V_N explicitly as

$$\begin{aligned} H_N &= \frac{1}{N} \left\langle \mathrm{Tr} \left[(\bar{z} - X^\dagger) \frac{1}{(z - X)(\bar{z} - X^\dagger) + |w|^2} \right] \right\rangle, \\ V_N &= \frac{1}{N} \left\langle \mathrm{Tr} \frac{\bar{w}}{(\bar{z} - X^\dagger)(z - X) + |w|^2} \right\rangle, \end{aligned} \tag{3.10}$$

and observe that $\lim_{w \to 0} H_N = \frac{1}{N} \left\langle \mathrm{Tr} \frac{1}{z - X} \right\rangle$ is the Green's function and $\lim_{w \to 0} V_N = 0$ unless $z = z_i$. The q-Green's function can be also expressed as a quaternionic "derivative" of the potential Φ_N:

$$\mathcal{G}_N(Q) = \delta_Q \Phi_N(Q), \qquad \delta_Q \equiv \begin{pmatrix} \partial_z & \partial_w \\ -\partial_{\bar{w}} & \partial_{\bar{z}} \end{pmatrix}, \tag{3.11}$$

which draws the frameworks closer on a formal level and from where the form of (3.8) is evident. Further developments in this direction proved to be successful in retrieving macroscopic eigenvalue densities of sums and products of matrices (Burda, Janik, and Nowak, 2011; Burda and Swiech, 2015).

Although this approach works on the level of Green's functions, we focus on an earlier step and discuss the role of the potential Φ_N. By (3.11), it is a basic object and will turn out to have a simple representation in the large matrix size limit. Before that we comment on two basic observables we aim at retrieving from the potential function. Firstly, by the Poisson equation (3.4), the spectral density reads:

$$\rho_N(z) = \frac{1}{\pi} \lim_{w \to 0} \partial_{\bar{z}} H_N = \frac{1}{\pi} \lim_{w \to 0} \partial_{\bar{z}z} \Phi_N(Q), \tag{3.12}$$

which is an exact result — no large matrix size limit needs to be taken. The second observable is the one-point eigenvector correlator defined as:

$$O_N(z) \equiv \frac{1}{N^2}\left\langle \sum_i O_{ii}\delta^{(2)}(z-z_i)\right\rangle, \tag{3.13}$$

where the eigenvector part $O_{ii} = \langle L_i|L_i\rangle \langle R_i|R_i\rangle$ comprises of left $\langle L_i|$ and right $|R_i\rangle$ eigenvectors. To remind a basic fact from algebra, a generic (i.e. not necessarily Hermitian) matrix X has two bi-orthogonal sets of eigenvectors :

$$\langle L_i| X = \langle L_i| z_i, \qquad X|R_i\rangle = z_i|R_i\rangle, \qquad \langle L_i|R_j\rangle = \delta_{ij}, \tag{3.14}$$

where these two sets coincide in the familiar case of $X = X^\dagger$. One consequence of imposing the bi-orthogonality condition is that among each family, the eigenvectors are not orthogonal ($\langle L_i|L_j\rangle \neq \delta_{ij}$ and $\langle R_i|R_j\rangle \neq \delta_{ij}$ in general). These objects were introduced in random matrix literature by Chalker and Mehlig (Chalker and Mehlig, 1998a; Chalker and Mehlig, 1998b) however in broader context O_N is a special observable of the Bell–Steinberger matrix (Savin and Sokolov, 1997; Fyodorov and Savin, 2012; Bell and Steinberger, 1966). As was found in (Janik et al., 1999), in the large N limit the eigenvector correlator is given by the module of the V_N part of the q-Green's function:

$$O(z) = \frac{1}{\pi}\lim_{N\to\infty}\lim_{w\to 0}|V_N|^2 = \frac{1}{\pi}\lim_{N\to\infty}\lim_{w\to 0}\partial_w\Phi_N(Q)\partial_{\bar{w}}\Phi_N(Q), \tag{3.15}$$

where $O(z) \equiv \lim_{N\to\infty} O_N(z)$ is the macroscopic correlator. On an operational level, it is sufficient to calculate the electrostatic potential Φ_N which will be addressed in Chapter 3.2. Before that, we discuss however the role of the w parameter. Although current treatment accentuates its regulatory character, it is simultaneously a natural way of coupling eigenvalues and eigenvectors in the potential Φ_N. To this end we make a similarity transformation of the potential function (3.7):

$$\mathrm{Tr}\log(Q-\mathcal{X}) = \mathrm{Tr}\log\begin{pmatrix} z-Z & -\bar{w}(S^\dagger S)^{-1} \\ wS^\dagger S & \bar{z}-Z^\dagger \end{pmatrix}, \tag{3.16}$$

with a similarity matrix S and diagonal matrix of eigenvalues Z defined by

$$S^{-1}XS = Z. \tag{3.17}$$

which is a matrix version of (3.14). The elements of off-diagonal matrices $(S^\dagger S)_{ij} = \langle R_i|R_j\rangle$ and $(S^\dagger S)^{-1}_{ij} = \langle L_i|L_j\rangle$ together comprise the eigenvector correlators O_{ii} of (3.13). Therefore, for $w \neq 0$ the potential Φ_N contains information on both eigenvalues Z and eigenvectors S. Since setting this parameter to zero from the beginning renders the potential independent of S and also makes the calculations intractable, we are led to believe that the degrees of freedom associated with w are of some importance.

3.1.2 Preliminary studies of dynamical GinUE

After a general discussion of the non-Hermitian realm we move on to present preliminary studies of the dynamical GinUE model of (3.1). As we argued before, we are particularly interested in the eigenvector dynamics. This discussion was largely presented in the work (Blaizot et al., 2016).

To gain more insight into the intertwined dynamics of the complex eigenvalues and eigenvectors of non-Hermitian matrices, let us perform some numerical experiments. We relegate to numerical simulations since the evolution equations for eigenvalues and eigenvectors are not available. Nonetheless, we inspect the evolution of the eigenvector matrix O_{ij} defined in (3.13) along with the eigenvalues to gain some insight.

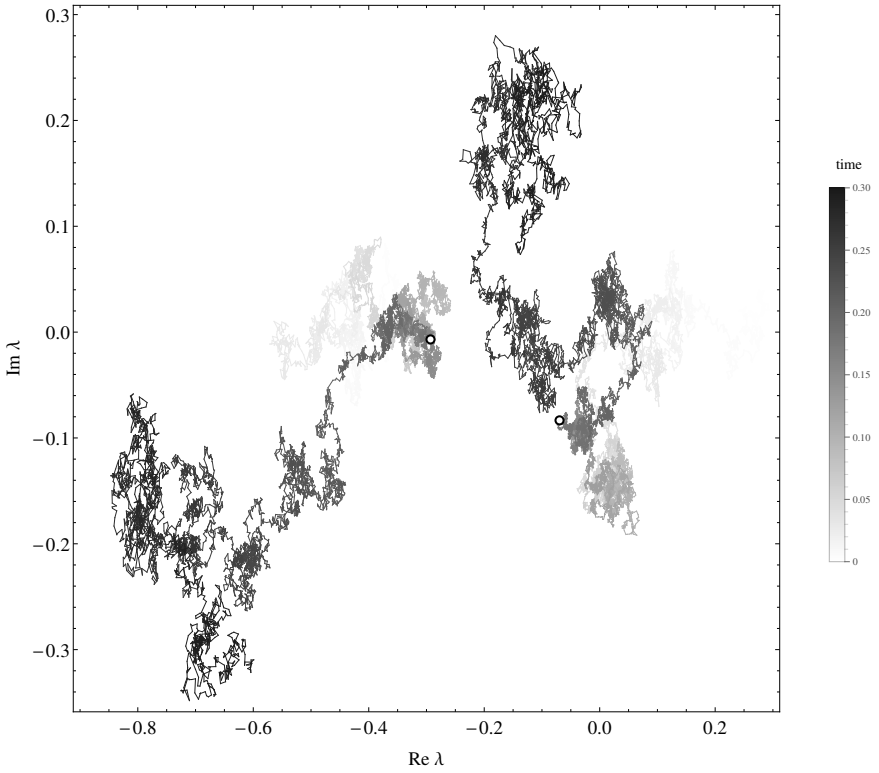

FIGURE 3.1: Numerical realization of a stochastic behaviour of $N = 2$ eigenvalues of a dynamical GinUE matrix evolving according to (3.1). Initial matrix $X_0 = \mathrm{diag}(0.3, -0.3)$ and the color of paths depict the progression of simulation time. Black edged white dots represent the position of eigenvalues at time $t = 0.1$ when the distance is minimized (see Fig. 3.2).

We focus on an example of an evolving dynamical GinUE matrix of size $N = 2$ starting from a diagonal matrix $X_0 = \mathrm{diag}(-0.3, 0.3)$. In Fig. 3.1, we observe the eigenvalues covering the complex plane in a diffusive manner. It is also expected that they repel each other. To perform a closer inspection (see Fig. 3.2), we plot three characteristics of their dynamics — the distance between the eigenvalues $|z_1 - z_2|$, the eigenvector matrix element $O_{11} = \langle L_1|L_1\rangle \langle R_1|R_1\rangle$ and the normalized jump $\Delta\lambda_1/(\Delta t)^{1/2}$ of the first eigenvalue, all as a function of simulation time t. We chose to ignore other eigenvector matrix elements $O_{22} = O_{11}$ and $O_{12} = 1 - O_{11}$ since they do not offer any additional information. The most interesting feature of this particular realization occurs around the time $t_c = 0.1$ of minimal eigenvalue distance (this precise moment is depicted by white dots on Fig. 3.2). We observe that, as the distance gets smaller, the element O_{11} blows up in a correlated manner. This is accompanied by an increase in the jump amplitude of the eigenvalue. This effect is generic also when matrix size $N > 2$. Analogous simulations (for which we relegate to our work (Blaizot et al., 2016))

show no such phenomenon happening in the standard dGUE case for which the distance between the eigenvalues also drives the evolution by the Vandermonde term. We therefore consider this effect as a qualitative demonstration of the co-dependence between the evolutions of eigenvalues and eigenvectors in this scenario which however is much stronger in the dGinUE than in dGUE.

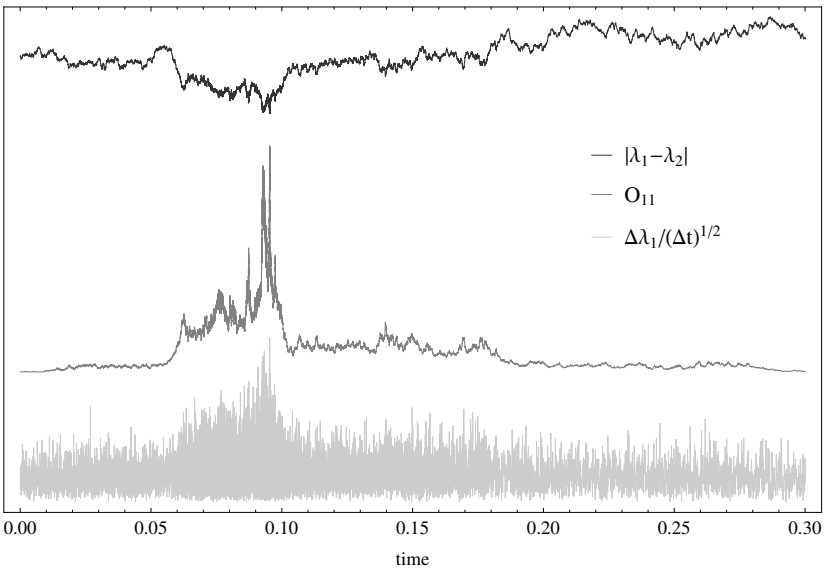

FIGURE 3.2: Time series of eigenvalue distance $|z_1 - z_2|$, eigenvector matrix element O_{11} and eigenvalue diffusion distance $\delta z_1 = \Delta\lambda_1/(\Delta t)^{1/2}$. Corresponding vertical axes are out of scale, we identify the time $t_c = 0.1$ to be of both minimal distance $|z_1 - z_2|$ and maximal values of O_{11} and δz_1. The correlation between the latter two latter observables is striking.

3.2 Dynamics of the averaged extended characteristic polynomial

Our aim is now to find a simple enough observable which will be related to the macroscopic quaternionic Green's function of (3.8). In the case of dGUE studied in Chapter 2.3, an averaged characteristic polynomial U_N of (2.26) served this purpose perfectly as it produced the standard Green's function of (2.27) via the Cole-Hopf transform (2.53). Analogously, in the non-Hermitian scenario of dGinUE we define an averaged extended characteristic polynomial (or AECP):

$$D_N(Q) \equiv \langle \det(Q - \mathcal{X}) \rangle = \left\langle \det\left((z - X)(\bar{z} - X^\dagger) + |w|^2 \right) \right\rangle . \tag{3.18}$$

We define effective potential φ_N and effective q-Green's function γ_N:

$$\varphi_N(Q) \equiv \frac{1}{N} \log D_N(Q), \tag{3.19}$$

$$\gamma_N(Q) \equiv \delta_Q \varphi_N(Q) \equiv \begin{pmatrix} h_N & v_N \\ -\overline{v_N} & \overline{h_N} \end{pmatrix}, \tag{3.20}$$

which coincide with the potential Φ_N and q-Green's function $\mathcal{G}_N$ in the large N limit. We present a diagram to make the above definitions and limiting procedures transparent:

object	exact		macroscopic		effective
potential	Φ_N (3.7)		Φ		φ_N (3.19)
q-Green's f. matrix	$\mathcal{G}_N$ (3.8)	$\xrightarrow{N\to\infty}$	$\mathcal{G}$	$\xleftarrow{N\to\infty}$	γ_N (3.20)
q-Green's f. elements	H_N, V_N (3.8)		H, V		h_N, v_N (3.20)
spectral density	ρ_N (3.5)		ρ		-
eigenvector corr.	O_N (3.13)		O (3.15)		-

We also include corresponding defining equations for the sake of clarity. The main message is that obtaining macroscopic formulae directly from the definitions (i.e. following the limiting procedure on the left) is as good as arriving from the effective counterparts (i.e. following the limiting procedure on the right). This approach is justified as long as the self-averaging property $\log\langle f\rangle \overset{N\to\infty}{\longrightarrow} \langle\log f\rangle$ is satisfied. Although we do not address this remarkable property, we have not found counterexamples to this rule. Main advantage is that all effective observables depend on the AECP, so it is enough to study its evolution.

3.2.1 Diffusion equation for the averaged extended characteristic polynomial

We consider the dynamical GinUE of (3.1) with time variable set as $N\tau = t$ (the physical origin of this rescaling is discussed in Chapter 2.3). The time argument τ is added to each object defined previously as they become dynamical.

In the paper (Burda et al., 2014) we write down the diffusion equation for the AECP as:

$$\partial_\tau D_N(Q;\tau) = \frac{1}{N}\partial_{w\bar{w}} D_N(Q;\tau), \tag{3.21}$$

which is exact in matrix size N and for the derivation we relegate to the paper (Burda et al., 2015). For an initial condition $D_N(Q;0) = \det\left((z-X_0)(\bar{z}-X_0^\dagger)+|w|^2\right)$ it is exactly solvable as

$$D_N(Q;\tau) = \frac{N}{\pi\tau}\int_C d^2w' \exp\left(-\frac{N}{\tau}|w-w'|^2\right) D_N(Q';0), \tag{3.22}$$

where $Q' = (z, w')$ is a quaternion of (3.6).

When compared with an analogous result of (2.28), now the diffusion happens in the auxiliary w variable. The z variable is in turn an independent parameter residing only in the initial condition. Although the w parameter was introduced in the potential (3.3) as a regulator, in the definition of the AECP its regulatory role unnecessary. Despite that, when the dynamics of D_N is considered, it becomes an crucial variable in which the diffusion of (3.21) happens. Furthermore, as was pointed out in (3.16), non-zero w variable makes D_N dependent on the eigenvectors. Therefore, a dynamical approach allows one to trace not only eigenvalues but also eigenvectors of the random matrix X_τ. To make the argument concrete, consider two different complex matrices X_1, X_2 with coinciding eigenvalues $Z_1 = Z_2$ (compare with (3.17) for the notation) but different eigenvectors $S_1 \neq S_2$. By (3.16), we find that $\det(Q-\mathcal{X}_1) \neq \det(Q-\mathcal{X}_2)$ but

at the same time $\det(Q-\mathcal{X}_1) \overset{w\to 0}{=} \det(Q-\mathcal{X}_2)$. In conclusion, eigenvectors matter in the non-Hermitian realm.

3.2.2 Hopf-Lax and Burgers' equations

Starting from (3.21), we derive equivalent equations for both the effective potential φ_N of (3.19) and the effective q-Green's function γ_N of (3.20) in the diagram:

name	equation
diffusion (3.21)	$\partial_\tau D_N = \frac{1}{N}\partial_{w\bar{w}} D_N$
	$\downarrow \frac{1}{N}\log$
Hopf-Lax (3.23)	$\partial_\tau \varphi_N = \frac{1}{N}\partial_{w\bar{w}}\varphi_N + \partial_w\varphi_N\partial_{\bar{w}}\varphi_N$
	$\downarrow \partial_w$, ∂_z
Burgers' (3.33) (3.34)	$\begin{cases}\partial_\tau v_N = \frac{1}{N}\partial_{w\bar{w}} v_N + \partial_w\lvert v_N\rvert^2 \\ \partial_\tau h_N = \frac{1}{N}\partial_{w\bar{w}} h_N + \partial_z\lvert v_N\rvert^2\end{cases}$

We move on to discussing advantages of considering each stage separately and the insights one gains by these equivalent forms.

Hopf-Lax equation. We move on to discuss the equation on the second row of the diagram:

$$\partial_\tau \varphi_N = \frac{1}{N}\partial_{w\bar{w}}\varphi_N + \partial_w\varphi_N\partial_{\bar{w}}\varphi_N, \tag{3.23}$$

where the argument of $\varphi_N(Q;\tau)$ was suppressed. In the large N limit we obtain a formula

$$\partial_\tau \Phi = \partial_w\Phi\partial_{\bar{w}}\Phi, \tag{3.24}$$

which we call the Hopf-Lax equation as its solution is given by the Hopf-Lax formula (Lax, 1957; Hopf, 1965):

$$\Phi(Q;\tau) = \max_{w'}\left(\Phi(Q';0) - \frac{|w-w'|^2}{\tau}\right). \tag{3.25}$$

This optimization task is solved by utilizing the radial symmetry $w = re^{i\alpha}$ and by noticing the convexity of (3.25). With these two properties, the macroscopic electrostatic potential Φ is given by an implicit equation

$$\begin{cases}\Phi(z,r;\tau) = \lim\limits_{N\to\infty}\frac{1}{N}\mathrm{Tr}\log\mathcal{M}(z,r_*) - \frac{(r-r_*)^2}{\tau}, \\ \lim\limits_{N\to\infty}\frac{r_*}{N}\mathrm{Tr}\mathcal{M}(z,r_*)^{-1} = \frac{r_*-r}{\tau},\end{cases} \tag{3.26}$$

where $\mathcal{M}(z,r) \equiv (z-X_0)(\bar{z}-X_0^\dagger)+r^2$. The macroscopic density $\rho(z;\tau) = \frac{1}{\pi}\partial_{z\bar{z}}\Phi(z,0;\tau)$ by the equation (3.12) and the macroscopic eigenvector correlator $O(z;\tau) = \frac{1}{4\pi}\lim_{r\to 0}\left(\partial_r \Phi(z,r;\tau)\right)^2$ by the formula (3.15) reads

$$\rho(z;\tau) = \rho^{(\mathrm{n})}(z;\tau) + \rho^{(\mathrm{nn})}(z;\tau), \tag{3.27}$$

$$\rho^{(\mathrm{n})}(z;\tau) = \lim_{N\to\infty}\frac{1}{N\pi}\frac{1}{\mathrm{Tr}\mathcal{M}^{-2}}\det\begin{pmatrix}\mathrm{Tr}(\bar{z}-X_0^\dagger)\mathcal{M}^{-2} & \mathrm{Tr}\mathcal{M}^{-2}r_* \\ -\mathrm{Tr}\mathcal{M}^{-2}r_* & \mathrm{Tr}(z-X_0)\mathcal{M}^{-2}\end{pmatrix}, \tag{3.28}$$

$$\rho^{(\mathrm{nn})}(z;\tau) = \lim_{N\to\infty}\frac{1}{N\pi}\mathrm{Tr}\left(\mathcal{M}^{-1}[\mathcal{M}^{-1};z-X_0](\bar{z}-X_0^\dagger)\right), \tag{3.29}$$

$$O(z;\tau) = \frac{1}{\pi\tau^2}r_*^2, \tag{3.30}$$

with condition for the maximizer r_* given by

$$\lim_{N\to\infty}\frac{1}{N}\mathrm{Tr}\mathcal{M}(z,r_*)^{-1} = \frac{1}{\tau}. \tag{3.31}$$

In the paper (Khoruzhenko, 1996), the formula $\rho^{(\mathrm{n})}$ was derived for the macroscopic spectral density valid for diagonal initial matrices X_0. An additional formerly unknown term $\rho^{(\mathrm{nn})}$ was found as a consequence of considering a triangular form of the matrix X_0. In complete agreement, this part vanishes $\rho^{\mathrm{nn}} = 0$ if X_0 is diagonal. Whether the general matrix X_0 is diagonal or not depends on its normality — normal matrices are diagonalizable by a unitary matrix (i.e. $S^\dagger = S^{-1}$ in (3.17)) whereas non-normal matrix can be only cast to a triangular form by a unitary transformation (by the Schur decomposition). The full equation (3.27) containing both terms $\rho^{(\mathrm{n})}$ and $\rho^{(\mathrm{nn})}$ is therefore valid for any initial matrix X_0. In Chapter 3.3.3 we consider an example with non-zero $\rho^{(\mathrm{nn})}$.

Equations (3.27) and (3.30) are valid inside the boundary given by

$$\lim_{N\to\infty}\frac{1}{N}\mathrm{Tr}\mathcal{M}(z,0)^{-1} = \frac{1}{\tau}, \tag{3.32}$$

which corresponds to $r_* = 0$. Outside this boundary, the correlators vanish $O(z;\tau) = 0$ and $\rho(z;\tau) = 0$ identically.

Burgers' equations. We consider a pair of equations for h_N and v_N comprising the q-Green's function of (3.10). We identify

$$\partial_\tau v_N = \frac{1}{N}\partial_{w\bar{w}}v_N + \partial_w|v_N|^2, \tag{3.33}$$

$$\partial_\tau h_N = \frac{1}{N}\partial_{w\bar{w}}h_N + \partial_z|v_N|^2, \tag{3.34}$$

as a pair of Burgers' equations where the arguments of $h_N(z,w;\tau)$ and $v_N(z,w;\tau)$ were suppressed. In the large N limit they read

$$\partial_\tau V = \partial_w|V|^2, \tag{3.35}$$

$$\partial_\tau H = \partial_z|V|^2, \tag{3.36}$$

where the first form an inviscid Burgers' equation in $2+1$ dimensions and the second is only a Burgers-like equation as it depends on the first (by $\partial_w H = \partial_z V$). We continue to discuss only the (3.35) in what follows. We set $\nu \equiv |V|$ and $r = |w|$ so that the 2+1

dimensional equation reduces to a Burgers' equation in 1+1 dimensions:

$$\partial_\tau \nu = \nu \partial_r \nu, \tag{3.37}$$

with argument $\nu(z, r; \tau)$ suppressed. It is an instance of first-order PDE of (2.56) and so the method of characteristics applies. The solution with an initial condition $\nu(z, r; 0)$ is given by

$$\nu = \nu(z, r + \tau\nu; 0). \tag{3.38}$$

From this solution we reconstruct the full $2 + 1$ solution of (3.35) as $V(z, w; \tau) = \frac{\bar{w}}{r}\nu(z, r; \tau)$ or deduce it from the Hopf-Lax formula (3.25) as:

$$V = V\left(z, w + \tau\bar{V}; 0\right). \tag{3.39}$$

We notice that the solution of (3.39) resembles a result obtained by the method of (complex) characteristics described in (2.56). Although both the z-space of (2.56) and w-space of (3.35) are two-dimensional, the underlying geometric structure is rather different — the former has a holomorphic structure whereas the latter is a vector-like equation. Despite this, both admit solutions in terms of characteristic lines. To conclude we cite the paper of (Bec and Khanin, 2007) where a theory of vector-like Burgers' equations of type (3.35) is reviewed in a systematic fashion.

3.3 Examples

To conclude we shortly describe three representative examples: the Ginibre evolution with vanishing initial matrix $X_0 = 0$, the spiric evolution for which $X_0 = \mathrm{diag}(\pm a)$ with an equal number of $\pm a$ eigenvalues and an evolution for a non-normal matrix $(X_0)_{ij} = \alpha\delta_{i,j-1}$.

3.3.1 Evolving Ginibre

We set the initial matrix $X_0 = 0$ for which the eigenvalue dynamics is depicted in Fig. 3.3. The density forms an "island" with size increasing in time.

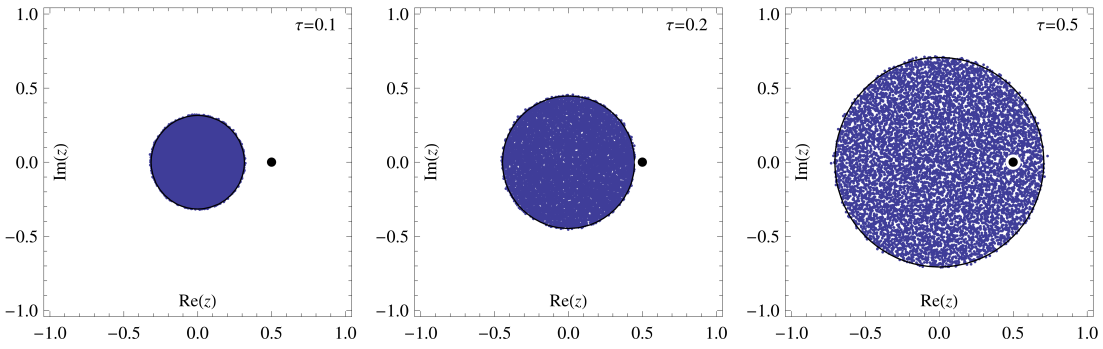

FIGURE 3.3: A numerical simulation of the spectral density with an initial matrix $X_0 = 0$ at time slices $\tau = 0.1, \tau = 0.2$ and $\tau = 0.5$ respectively, an ensemble of 6 matrices of size $N = 1500$. Black curves are the large N spectral boundaries $\tau = |z|^2$ and the dot indicates an arbitrary spectator position z where the evolution is probed.

To start, the boundary equation (3.32) reads $\tau = |z|^2$ and defines a circle of radius $\sqrt{\tau}$ inside which the eigenvalues reside. The maximizer r_* of (3.31) is either 0 for $\tau < |z|^2$ or $\sqrt{\tau - |z|^2}$ for $\tau > |z|^2$. With this information, the macroscopic eigenvector correlator (3.30) and eigenvalue density (3.27) are obtained:

$$O(z;\tau) = \begin{cases} 0 & \text{for} \quad \tau < |z|^2 \\ \frac{1}{\tau^2\pi}(\tau - |z|^2) & \text{for} \quad \tau > |z|^2 \end{cases}, \tag{3.40}$$

$$\rho(z;\tau) = \begin{cases} 0 & \text{for} \quad \tau < |z|^2 \\ \frac{1}{\pi\tau} & \text{for} \quad \tau > |z|^2 \end{cases}. \tag{3.41}$$

Although this approach most straightforwardly produce well-known formulae, analysis at the level of Burgers' equations is interesting from a conceptual point of view. Here we only point out the most profound consequences of the dynamics of characteristics and refer to (Burda et al., 2015) for the details. As was found in Hermitian and unitary cases (Blaizot and Nowak, 2008; Blaizot and Nowak, 2009a), the shocks of an inviscid Burgers' equation (2.55) are generally identified with the edges of the spectral densities. In present setting however, the situation is drastically different — the shock-lines of (3.37) coincide exactly with the locus where the correlators ρ or O are non-zero. Therefore, it does not matter whether we probe the edge of the density (i.e. the circle of radius $\sqrt{\tau}$) or 'observe" the evolution from inside the bulk as in the rightmost plot on Fig. 3.3 — we are constantly "in shock" from a fluid perspective.

Universal results. So far we have discussed the macroscopic limit $N \to \infty$, a natural question now is to ask about finite N results and its universal character. Although our method is not tailored to find finite N results of spectral densities or eigenvector correlators in general, diffusion equation (3.21) for the AECP is exact for any matrix size N.

We look at the AECP for an initial condition $D_N(z, r; 0) = (|z|^2 + r^2)^N$. We use the radial symmetry, set $|w| = r$ and we obtain

$$D_N(z, r; \tau) = \frac{2N}{\tau} \int_0^\infty r' e^{-\frac{N}{\tau}(r^2+r'^2)} I_0\left(\frac{2Nrr'}{\tau}\right) (|z|^2 + r'^2)^N dr'. \tag{3.42}$$

On the complex plane z and for $r = 0$ this integral significantly simplifies and we approximate it via the saddle point method. There are three saddle points $r'_0 = 0, r'_\pm = \pm\sqrt{\tau - |z|^2}$ so when τ approaches $|z|^2$, the two points $r'_\pm$ coalesce for $\tau_c = |z|^2$ and at $r' = 0$. This collision happens near the edge of the spectrum (compare with (3.41)) just as was the case in the analysis of the ACP/AICP in the dGUE model conducted in Chapter 2.3.1. Near the critical value τ_c we introduce parameters $\theta = r'N^{1/4}, \eta = N^{1/2}(|z| - \sqrt{\tau})$. One finds that in the large N limit, the AECP has an asymptotic form

$$D_N(|z| = \sqrt{\tau} + \eta N^{-1/2}, 0; \tau) \sim \tau^N \sqrt{\frac{N\pi}{2}} e^{\frac{\eta}{\tau}(2\sqrt{N\tau}+\eta)} \operatorname{erfc}\left(\sqrt{\frac{2}{\tau}}\eta\right), \tag{3.43}$$

where erfc is the complementary error function.

The error function behaviour is of the same form as in the finite size expression for the eigenvalue density function which is not surprising as in the GinUE models these two are proportional to each other (Akemann and Vernizzi, 2003). The proportionality function found in our paper (Burda et al., 2015) turned out to exactly cancel the spurious term in (3.43) and thus reproduced the well-known erfc universal behaviour near

the edge

$$\rho(\eta;\tau) \sim \frac{1}{2\pi\tau}\mathrm{erfc}\left(\sqrt{\frac{2}{\tau}}\eta\right), \tag{3.44}$$

where no dependence on N prevails.

3.3.2 Evolution of the spiric section

We turn to the evolution of a diagonal matrix $X_0 = \mathrm{diag}(\pm a)$ with the same number of $\pm a$ eigenvalues. In this example depicted on Fig. 3.4, the $N/2$ eigenvalues start concentrated at two separate locations near $\pm a$, expand in time and finally join at some critical moment τ_c. We find the boundaries from the equation (3.32) as:

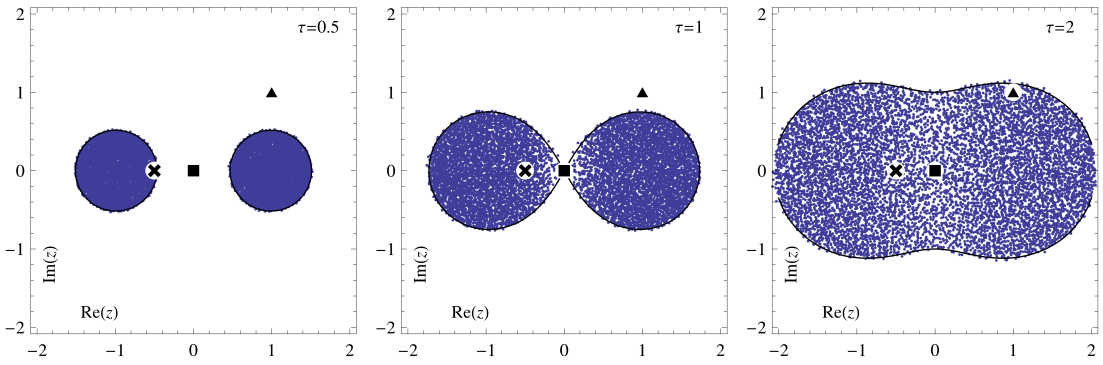

FIGURE 3.4: The spectral density dynamics for an initial matrix $X_0 = \mathrm{diag}(1,...,1,-1,...,-1)$ before, at and after the critical time $\tau_c = 1$, the ensemble consisted of 6 matrices of size $N = 1500$. The cross, square and triangle denote three observers useful in the analysis of the evolution.

$$\tau(|a|^2 + |z|^2) = |a^2 - z^2|^2, \tag{3.45}$$

and identify it as a spiric section[1], the corresponding area is symbolically denoted as $\mathcal{S}$. The maximizer of (3.31) is $r_* = 0$ if $z \notin \mathcal{S}$ and $r_* = \sqrt{\frac{\tau}{2} - |a|^2 - |z|^2 + \frac{1}{2}S_a}$ for $z \in \mathcal{S}$. We introduce the variables $S_a = \sqrt{\tau^2 + 4Z_a^2}$, $Z_a = \bar{z}a + z\bar{a}$ and we plug the maximizer into (3.30) and (3.27) to obtain the correlators:

$$O(z;\tau) = \begin{cases} 0 & \text{for} \quad z \notin \mathcal{S} \\ \frac{1}{\tau^2\pi}\left(\frac{\tau}{2} - |a|^2 - |z|^2 + \frac{1}{2}S_a\right) & \text{for} \quad z \in \mathcal{S} \end{cases}, \tag{3.46}$$

$$\rho(z;\tau) = \begin{cases} 0 & \text{for} \quad z \notin \mathcal{S} \\ \frac{S_a(2-\tau|a|^2)+\tau^2|a|^2}{2\pi\tau Z_a^2 S_a} & \text{for} \quad z \in \mathcal{S} \end{cases}. \tag{3.47}$$

Although the most tangible results are the formulae for the spectral density and eigenvector correlators, it is perhaps more interesting that, at the level of Burgers' equations, this example and the one discussed in Chapter 3.3.1 do not differ in a qualitative way. The only difference is the place where the shock-line begins — in previous case it was

[1] A spiric section is defined as the curve of intersection of a torus and a plane parallel to its rotational symmetry axis

the time $\tau = |z|^2$ whereas in this example the shock-line starts at $\tau_c = \frac{|a^2-z^2|^2}{|a|^2+|z|^2}$. The dynamics remain essentially the same which is corroborated by the asymptotic analysis conducted next.

Universal results. We complete the discussion of the spiric case by deriving a finite N formula for the AECP of (3.22) with the initial condition $D_N(z,r;0) = (|z-a|^2 + r^2)^{N/2}(|z+a|^2+r^2)^{N/2}$:

$$D_N(z,r;\tau) = \frac{2N}{\tau}\int_0^\infty r' e^{-\frac{N}{\tau}(r^2+r'^2)} I_0\left(\frac{2Nrr'}{\tau}\right) \times \\ \times (r'^2+|z-a|^2)^{\frac{N}{2}}(r'^2+|z+a|^2)^{\frac{N}{2}} dr'. \tag{3.48}$$

The most interesting case would be when two eigenvalue "islands" join into one. This happens near the origin $z = 0$ for τ close to the collision time τ_c. We conduct the calculation for $a = 1$ in which case $\tau_c = 1$. We zoom into the region where the merging of saddle points happens and set $\theta = rN^{1/4}, \eta = zN^{1/4}$ and $t = N^{1/2}(\tau - 1)$. An asymptotic formula then reads

$$D_N(\eta N^{-1/4}, 0; 1+tN^{-1/2}) \sim \sqrt{\frac{\pi}{128N}} e^{-\frac{\sqrt{N}}{2}((\eta+\bar\eta)^2-2|\eta|^2)}(\eta+\bar\eta)^4 \times \\ \times \operatorname{erfc}\left[\frac{1}{\sqrt{2}}\left(|\eta|^2 - (\eta+\bar\eta)^2 - t\right)\right]. \tag{3.49}$$

Although in this case the AECP and spectral density are not proportional to each other, we still find out an error function in (3.49).

Firstly, the lack of a truly new universal function should be expected as no new (when compared to the example of Chapter 3.3.1) shock structure was present on the level of Burgers' equation. Moreover, a physical explanation of this fact is available. Novel universal behaviour happens as macroscopic groups of eigenvalues (i.e. the bulks) collide into each other *en masse* exerting huge "tension" on individual particles. Perhaps the most basic example is the Pearcey integral (2.37) and (2.40) arising in the dGUE case as the collision of two such bulks happens in Fig 2.1. In the present setting of dGinUE however, the situation is different. Although the interaction between the eigenvalue "islands" is of similar fashion, each particle lives on a two-dimensional plane and thus is free to escape upon any "tension" applied by the neighbouring bulk. This is not the case however when eigenvalues are bound to a line as in Fig. 2.1 — then they are stuck between a rock and a hard place. Hence, no necessary tension in dGinUE happens and no new critical behaviour is found.

3.3.3 Non-normal Ginibre case

Last example probes the non-normal regime. To this end we initiate the evolution with a matrix $(X_0)_{ij} = \alpha\delta_{i,j-1}$ which has all eigenvalues equal to 0 and non-trivial eigenvector information. Three snapshots of the evolution are presented in Fig. 3.5. Eigenvalues initially concentrated at zero instantaneously (from the macroscopic point of view) blow up to a circle of radius $|\alpha|$. After that, the density grows into an annulus and after finite time $\tau_0 = |\alpha|^2$ it forms a disc as the inner radius shrinks to zero. The radii of annulus are given by the equation (3.32) as

$$|z|_\pm = \sqrt{|\alpha|^2 \pm \tau}, \tag{3.50}$$

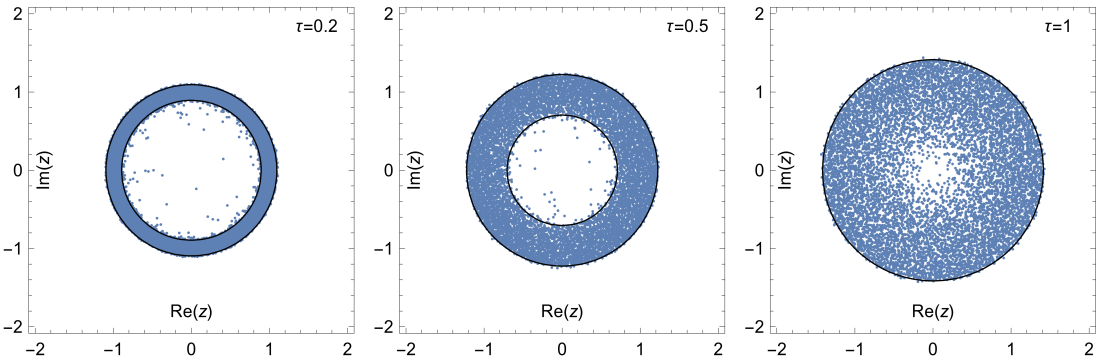

FIGURE 3.5: Evolution of the spectral density for non-normal initial condition $(X_0)_{ij} = \delta_{i,j-1}$, with time snapshots at $\tau = 0.2$, $\tau = 0.5$ and $\tau = 1$, the ensemble consisted of 6 matrices of size $N = 1500$.

where the inside of the annulus is denoted symbolically as $\mathcal{A}$. The maximizer parameter of (3.31) read $r^* = 0$ outside of $\mathcal{A}$ and $r^* = \sqrt{T_\alpha - |z|^2 - |\alpha|^2}$ inside $\mathcal{A}$ with the notation $T_\alpha = \sqrt{\tau^2 + 4|\alpha|^2|z|^2}$. Again, we find the correlators by the equations (3.30) and (3.27) as

$$O(z;\tau) = \begin{cases} 0 & \text{for} \quad z \notin \mathcal{A} \\ \frac{1}{\pi\tau^2}\left(T_\alpha - |z|^2 - |\alpha|^2\right). & \text{for} \quad z \in \mathcal{A} \end{cases}, \tag{3.51}$$

$$\rho(z;\tau) = \begin{cases} 0 & \text{for} \quad z \notin \mathcal{A} \\ \frac{1}{\pi\tau}\left(1 - \frac{|\alpha|^2}{T_\alpha}\right) & \text{for} \quad z \in \mathcal{A} \end{cases}. \tag{3.52}$$

Above expressions reduce to the case of Ginibre discussed in Chapter 3.3.1 when $\alpha \to 0$. To arrive at these results for non-normal initial conditions, a little bit more algebra is needed as $\mathcal{M}$ of is no longer diagonal. Detailed calculations are given in the source paper (Burda et al., 2015).

Universal results. Let us shortly discuss finite N effects for the AECP. We are looking for an universal behaviour near the origin for $\tau \to 0$ where a seemingly instantaneous inflation of eigenvalues from zero to a finite radius $|\alpha|$ happens. To this end we write down the AECP as:

$$D_N(z, r; \tau) = \frac{2N}{\tau} \int_0^\infty r' e^{-\frac{N}{\tau}(r^2 + r'^2)} I_0\left(\frac{2Nrr'}{\tau}\right) \det \mathcal{M}(z, r') dr', \tag{3.53}$$

with the determinant $\det \mathcal{M}$ given by

$$\det \mathcal{M} = \frac{1}{\Delta}\left(d(a_+^N - a_-^N) - |b|^2(a_+^{N-1} - a_-^{N-1}\right), \tag{3.54}$$

where $a = |z|^2 + r^2 + |\alpha|^2$, $b = -\bar{z}\alpha$, $\Delta = \sqrt{a^2 - 4|b|^2}$ and $a_\pm = \frac{1}{2}(a \pm \Delta)$. We consider the following scaling around the origin $\theta = N^{7/6}, x = |z|N^{1/6}, t = N^{4/3}\tau$ and $x = |\alpha|N^{1/6}$. Around these points we find an asymptotic formula:

$$D_N(xN^{-1/6}, 0; tN^{-4/3}) \sim c_N \frac{t}{2} + c_N \frac{t\sqrt{\pi t}}{4x} \exp\left(\frac{t}{4x^2}\right) \operatorname{erfc}\left(-\frac{\sqrt{t}}{2x}\right). \tag{3.55}$$

where we have skipped the N dependent pre-factor c_N for clarity. A non-perturbative term divergent near $x = 0$ is present — as it is absent in the same scaling of a pure Ginibre, we identify it as a mark of the non-normality. Not surprisingly, it has indeed a delta-like peak at zero, spreading as we vary the time variable which corresponds to the movement of eigenvalues as time increases. The picture is however not full as we have also scaled the $|\alpha|$ parameter to be near zero.

3.4 Dynamical GinUE in a quadratic potential

Dynamical non-Hermitian models discussed in Chapter 3.1 are diffusing freely in both the matrix entries and its eigenvalues — it is evident from the numerical studies of Figures 3.3, 3.4 and 3.5 where the eigenvalue clouds always "dissolve" into infinity as $\tau \to \infty$. Therefore no stationary $\tau \to \infty$ limit exists just as in the case of free dGβE models discussed in Chapter 2.4. To re-introduce a stationary limit, we consider a model with a confining quadratic potential as a non-Hermitian matrix generalization of the Ornstein-Uhlenbeck process which we abbreviate as dGinβE-OU. Such family is exactly a non-Hermitian counterpart of the confined dGβE (or dGβE-OU) discussed in Chapter 2.4 and arranged in Tab. 2.2. Accordingly, we write down in Tab. 3.2 an additional confining term needed besides the free dGinβE spatial operator of Tab. 3.1. This part is based on the work (Blaizot et al., 2016).

β	$\delta\Delta_X$ operator
1	$aN \sum_{i,j=1}^{N} \partial_{x_{ij}} x_{ij}$
2	$aN \sum_{d=1}^{2} \sum_{i,j=1}^{N} \partial_{x_{ij}^{(d)}} x_{ij}^{(d)}$
4	$aN \sum_{d=1}^{4} \sum_{i,j=1}^{N} \partial_{x_{ij}^{(d)}} x_{ij}^{(d)}$

TABLE 3.2: Additional term $\delta\Delta_X$ to the spatial operator Δ_X of (3.1) introducing a confining quadratic potential of Ornstein-Uhlenbeck type (or simply dGβE-OU). Real and complex matrices are N dimensional, symplectic case is of size $2N$. The parameter a controls the potential width.

From now on we consider the $\beta = 2$ case and read off the SFP equation (3.1) from the second rows of Tables 3.1 and 3.2 as:

$$\partial_t P_t\left(X|X_0\right) = \sum_{i,j=1}^{N} \left[\frac{1}{4}\left(\partial_{x_{ij}}^2 + \partial_{y_{ij}}^2\right) + Na\left(\partial_{x_{ij}} x_{ij} + \partial_{y_{ij}} y_{ij}\right)\right] P_t\left(X|X_0\right). \tag{3.56}$$

With $\tau = Nt$ and up to an irrelevant normalization constant D'_t, a solution to (3.56) with an initial condition $P_0(X|X_0) = \delta(X - X_0)$ reads

$$P_\tau\left(X|X_0\right) = \frac{1}{D'_\tau} \exp\left(-\frac{2Na}{1 - e^{-2a\tau}} \mathrm{Tr}|X - X_0 e^{-a\tau}|^2\right). \tag{3.57}$$

For short times $\tau \sim 0$, the solution (3.57) reduces to the free dGinUE model of (3.2) whereas for large times $\tau \to \infty$ it is asymptotically approaching a stationary Ginibre Ensemble $\lim_{\tau\to\infty} P_\tau \sim \exp\left(-2Na\mathrm{Tr}|X|^2\right)$.

We consider an evolution of the AECP of (3.18) where the dynamics of X is governed by the SFP equation (3.56). We write down an exact equation for the AECP:

$$\partial_\tau D_N(Q;\tau) = \frac{1}{N}\partial_{w\bar{w}} D_N(Q;\tau) - 2NaD_N(Q;\tau) + a\mathrm{d}D_N(Q;\tau), \tag{3.58}$$

with the operator $\mathrm{d} \equiv z\partial_z + \bar{z}\partial_{\bar{z}} + w\partial_w + \bar{w}\partial_{\bar{w}}$. For the derivation we relegate interested reader to the source paper (Blaizot et al., 2016). In the limit of $a \to 0$ we recover the free case (3.21). We introduce the Lamperti transformation $(Q,\tau) \to (Q',\tau')$:

$$\begin{aligned} &Q' = e^{a\tau}Q, \quad \tau' = \frac{1}{2a}\left(e^{2a\tau} - 1\right), \\ &D_N(Q;\tau) = (1+2a\tau')^{-N} D'_N(Q';\tau'). \end{aligned} \tag{3.59}$$

which leads to the free diffusion in the (Q',τ') variables:

$$\partial_{\tau'} D'_N(Q';\tau') = \frac{1}{N}\partial_{w'\bar{w}'} D'_N(Q';\tau'). \tag{3.60}$$

Details on arriving at the transformation (3.59) are given in Appendix B.2. We conclude that, similarly to the dGUE case of Chapter 2.4, the presence of a confining potential does not produce novel phenomena and therefore it is not an essential feature of the dynamics.

3.5 Exact spectral densities

Despite the success of the dynamical approach in understanding the non-Hermitian models better, most of the results reported in previous chapters were found in the macroscopic limit. It is therefore natural to ask about finite matrix size (or exact) results in the dynamical GinUE ensemble. In particular, the exact form of spectral density (3.5) remained out of our reach. To achieve such goal, we move outside the dynamical framework discussed so far and apply a powerful supersymmetric technique to this problem. In the work (Grela and Guhr, 2016) we study a noise-plus-structure model suitable for inference tasks — let M be a matrix of the form:

$$M = S + LXR, \tag{3.61}$$

where S is a fixed matrix and $L, R > 0$ are diagonal positive definite covariance matrices. The matrix X is the source of noise drawn typically from a static Ginibre Unitary Ensemble. The matrix S is called a source and is interpreted as the signal/information matrix of the system in study. We add a structured noise LXR as every real-world data is contaminated, and only the resulting matrix M is attainable by experiment.

Instances of the model of (3.61) with differently interpreted S, L, X and R are abundant in the literature. Absence of any structure means setting $S = 0$ and $L = R = 1$ and reduces it to a standard random matrix theory models of pure randomness. By setting $S = e^{-\tau}H_0$, $LR = \sqrt{1-e^{-2\tau}}$ and with both H_0 and X being random matrices drawn from the GUE, we model an ensemble of charged spin-less particles interacting with a strong external magnetic field proportional to the τ parameter (Lenz and Haake, 1990). If the matrix S is drawn from the GUE, $LR = -i\pi$, and $X = W^\dagger W$ with W drawn from a GinUE, resulting model describes quantum chaotic scattering in open cavities as a random interaction between the cavity and its surroundings (Fyodorov and Sommers, 1997).

Purely non-Hermitian models of the type (3.61) appear as toy-models of neuronal networks (Rajan and Abbott, 2006; Luca, Ricciardi, and Vasudevan, 1970; Amir, Hatano, and Nelson, 2016). Here, M represents the neuronal adjacency matrix and we begin with setting $S = 0, L = R = 1$. In this context however, an additional constraint is needed — each matrix row must be either purely negative or purely positive which reflects the Dale's Law of neuronal behaviour. Moreover, a recent paper (Ahmadian, Fumarola, and Miller, 2015) argued that also the S, L and R matrices in the model (3.61) might be of significance.

In our paper we continue and expand on the works of (Khoruzhenko, 1996; Hikami and Pnini, 1998; Ahmadian, Fumarola, and Miller, 2015) where the model of (3.61) was studied either in the large N limit or for $L = R = 1$ and diagonal S.

3.5.1 Generating ratio

Let X of (3.61) be an $N \times N$ matrix drawn from a jPDF of the standard GinUE,

$$P(X) = \frac{1}{\tilde{C}} \exp\left(-n \mathrm{Tr} X^\dagger X\right), \tag{3.62}$$

where $\tilde{C} = (\pi/n)^{N^2}$ is the normalization constant. The source matrix S of (3.61) is in the most general form given by $S = D + T$ where D is diagonal and T is strictly upper triangular as a consequence of the Schur decomposition. When $T = 0$, the source matrix is called normal, otherwise it is non-normal.

A basic statistical quantity characterizing the model (3.61) is the spectral density of (3.5) where averaging is done over (3.62). To find the spectral density, we define the averaged over (3.62) extended ratio of determinants

$$\mathcal{R}_{L,R}(Z, V) \equiv \left\langle \frac{\det(Z - \mathcal{M})}{\det(V - \mathcal{M})} \right\rangle_P \tag{3.63}$$

where the $2N \times 2N$ block matrices read

$$\mathcal{M} \equiv \begin{pmatrix} 0 & M \\ M^\dagger & 0 \end{pmatrix}, \quad Z \equiv \begin{pmatrix} L^2 w & z\mathbf{1}_N \\ \bar{z}\mathbf{1}_N & -R^2 \bar{w} \end{pmatrix}, \quad V \equiv \begin{pmatrix} L^2 u & v\mathbf{1}_N \\ \bar{v}\mathbf{1}_N & -R^2 \bar{u} \end{pmatrix}, \tag{3.64}$$

where $\mathbf{1}_N$ denotes the $N \times N$ unit matrix and the matrices Z and V depend on the complex variables z, u, v and w forming quaternions akin to (3.6). For us, the main reason of introducing (3.63) lies in the following exact formula for the spectral density (3.5):

$$\rho_N(z) = -\frac{1}{N\pi} \lim_{w \to 0} \frac{\partial}{\partial \bar{z}} \lim_{V \to Z} \frac{\partial}{\partial v} \mathcal{R}_{L,R}(Z, V) \tag{3.65}$$

introduced in (Fyodorov, Khoruzhenko, and Sommers, 1997). The average of (3.63) is a tedious task and was done by the supersymmetry method in the paper (Grela and Guhr, 2016). The resulting formula is equal to:

$$\begin{aligned} \mathcal{R}_{L,R} = & \frac{4i}{\pi} \int_{-\infty}^{\infty} dg_- \int_0^{\infty} df g_- f e^{-n(g_-^2 + f^2 + |w|^2 - |u|^2)} \times \\ & \times I_0(2nf|w|) K_0(2in|u|g_-) G\Big[g_1 + (n - g_2)(n - g_3) + g_4\Big], \end{aligned} \tag{3.66}$$

where $g_- \equiv g - i\epsilon$, $\epsilon > 0$ and I_0 (K_0) is the modified Bessel function of the first (second) kind. Furthermore, the integrand consists of

$$g_1 = f^2 g_-^2 \operatorname{Tr}\left[\mathbf{P}_v \mathbf{Q}_z\right] \operatorname{Tr}\left[\mathbf{P}'_v \mathbf{Q}'_z\right], \quad g_2 = \operatorname{Tr}\left[\Omega_z \Gamma_v \mathbf{P}_v \mathbf{Q}_z\right], \quad g_3 = \operatorname{Tr}\left[\Omega_v \Gamma_z \mathbf{Q}_z \mathbf{P}_v\right],$$
$$G = \frac{\det(-f^2 \mathbf{1}_N - \Gamma_z \Omega_z)}{\det(g_-^2 \mathbf{1}_N - \Gamma_v \Omega_v)}, \quad g_4 = f^2 \operatorname{Tr}\left[\Omega_v \mathbf{Q}'_z \Gamma_v \mathbf{P}_v \mathbf{Q}_z \mathbf{P}_v\right] + g_-^2 \operatorname{Tr}\left[\Omega_z \mathbf{P}'_v \Gamma_z \mathbf{Q}_z \mathbf{P}_v \mathbf{Q}_z\right],$$

with the additional notation:

$$\begin{aligned} \Omega_x &\equiv R^{-2}(\bar{x}\mathbf{1}_N - S^\dagger), & \Gamma_x &\equiv L^{-2}(x\mathbf{1}_N - S), \\ \mathbf{P}_v &\equiv (g_-^2 \mathbf{1}_N - \Omega_v \Gamma_v)^{-1}, & \mathbf{P}'_v &\equiv (g_-^2 \mathbf{1}_N - \Gamma_v \Omega_v)^{-1}, \\ \mathbf{Q}_z &\equiv (-f^2 \mathbf{1}_N - \Omega_z \Gamma_z)^{-1}, & \mathbf{Q}'_z &\equiv (-f^2 \mathbf{1}_N - \Gamma_z \Omega_z)^{-1}. \end{aligned}$$

Although the result of (3.66) is complex, we will find particular cases where it simplifies considerably by identifying a few building blocks akin to those considered in Chapter 2.3.1.

3.5.2 Particular cases

We restrict ourselves to reporting on three wide classes of problems considered in detail in (Grela and Guhr, 2016) where the ratio formula (3.66) simplify:

1. normal source S and variance matrices L, R diagonal and otherwise arbitrary,
2. non-normal source S of rank one and trivial variance matrices $L = R = 1$,
3. the spectral density of M^{-1}.

Normal S and arbitrary L, R

In this case all structure matrices L, R and S are diagonal,

$$\begin{aligned} S &= \operatorname{diag}(\underbrace{s_1, ..., s_1}_{u_1}, \underbrace{s_2, ..., s_2}_{u_2}, \underbrace{..., s_x}_{...u_x}), \\ L &= \operatorname{diag}(\underbrace{l_1, ..., l_1}_{v_1}, \underbrace{l_2, ..., l_2}_{v_2}, \underbrace{..., l_y}_{...v_y}), \\ R &= \operatorname{diag}(\underbrace{r_1, ..., r_1}_{w_1}, \underbrace{r_2, ..., r_2}_{w_2}, \underbrace{..., r_z}_{...w_z}), \end{aligned}$$

with three sets of multiplicities u_i, v_i, w_i. Here, x, y, z are the numbers of different entries in the structure matrices L, R and S, respectively, thereby defining the sizes of the sets. The multiplicities in each set add up to N. Since now the integrand (3.66) depends only on the products $(\Omega_x)_{ii}(\Gamma_y)_{ii}$, we introduce a new source matrix:

$$\alpha_{xy} \equiv \Omega_x \Gamma_y = (LR)^{-2}(\bar{x}\mathbf{1}_N - S^\dagger)(y\mathbf{1}_N - S), \tag{3.67}$$

which is accompanied by a merged multiplicity vector $\vec{n}$ which we extract out of $\vec{u}$,$\vec{v}$ and $\vec{w}$ in the following way: we first form the multiplicity vectors $\vec{u} = (u_1, ..., u_x)$, $\vec{v} = (v_1, ..., v_y)$ and $\vec{w} = (w_1, ..., w_z)$ corresponding to the matrices S, L and R, respectively. Then the vectors $\vec{u}$ ($\vec{v}, \vec{w}$) are graphically represented by a column of N points ordered in x (y, z) groups according to the multiplicities u_i (v_i, w_i). We refer to the first and last points in each group as a boundary. The multiplicity vector $\vec{n} = (n_1, ..., n_k)$ is then

constructed as a vector which has a boundary whenever *at least* one of the vectors $\vec{u}, \vec{v}$ and $\vec{w}$ has one. From now on we only use the merged vector $\vec{n}$. We introduce the dimension $d(\vec{n})$ of the vector $\vec{n}$ as the number of differing groups. We also introduce the length $|\vec{n}| = \sum_{i=1}^{d(\vec{n})} n_i$. With the help of $\vec{n}$, the generating ratio (3.66) is rewritten as

$$\begin{aligned}\frac{1}{C}\mathcal{R}_{L,R} = i_{\vec{n}} j_{\vec{n}} - \sum_{i=1}^{d(\vec{n})} \frac{n}{n_i}\left(\alpha_{zv}^i + \alpha_{vz}^i + \frac{N}{n}\right) i_{\vec{n}-\vec{e}_i} j_{\vec{n}+\vec{e}_i} + \\ + \sum_{i,j=1}^{d(\vec{n})} \frac{n^2 \alpha_{zv}^i}{n_i n_j}\left[\left(\alpha_{vz}^j - \alpha_{vz}^i\right) i_{\vec{n}-\vec{e}_i-\vec{e}_j} j_{\vec{n}+\vec{e}_i+\vec{e}_j}\right] + \\ + \sum_{i,j=1}^{d(\vec{n})} \frac{n}{n_j}\left[\alpha_{vv}^i i_{\vec{n}-\vec{e}_j} j_{\vec{n}+\vec{e}_i+\vec{e}_j} + \alpha_{zz}^i i_{\vec{n}-\vec{e}_i-\vec{e}_j} j_{\vec{n}+\vec{e}_j}\right],\end{aligned} \tag{3.68}$$

where α_{xy}^i is the i–th element of the diagonal matrix (3.67), $C = \prod_{i=1}^{d(\vec{n})} n_i$, and the $\vec{e}_i$'s are k–dimensional unit vectors in the i–th direction. The result (3.68) contains two basic elements referred to as the fermionic ($i_{\vec{n}}$) and bosonic ($j_{\vec{n}}$) building blocks. The former is given by

$$i_{\vec{m}}(z,w) = \frac{e^{-n|w|^2}}{\prod_{i=1}^{d(\vec{m})} m_i!}\int_0^\infty d\rho e^{-\rho} I_0(2\sqrt{n\rho}|w|)\prod_{i=1}^{d(\vec{m})}\left(\rho + n\alpha_{zz}^i\right)^{m_i}, \tag{3.69}$$

where we set $i_{\vec{m}} = 0$ if some element of the multiplicity vector $\vec{m}$ is negative. The bosonic counterpart reads

$$j_{\vec{m}}(v,u) = \frac{\prod_{i=1}^{d(\vec{m})}(m_i-1)!}{2\pi i}\oint_{\Gamma_s} dp \sum_{k=0}^{\infty}\frac{U_{k+1,1}(n|u|^2)p^k}{\prod_{i=1}^{d(\vec{m})}\left(p + n\alpha_{vv}^i\right)^{m_i}}, \tag{3.70}$$

where the contour Γ_s encircles counter-clockwise all of the sources $-n\alpha_{vv}^i$. The function $U_{a,b}(z) = U(a,b,z)$ is the Tricomi confluent hypergeometric function. Details of arriving at (3.70) are provided in the source paper (Grela and Guhr, 2016).

We notice that in the formula (3.65), the parameters u and w serve as regulators. It is desirable to set them to zero before computing the derivatives. Even though this does not pose a problem for the fermionic block (3.69), it produces infinities in the bosonic block (3.70). To control this singular behaviour we follow a "regularization" procedure and introduce a new bosonic block finite in the $u \to 0$ limit:

$$j_{\vec{m}}(v,u) \to \tilde{j}_{\vec{m}}(v,u). \tag{3.71}$$

The corresponding regularized ratio is then defined as

$$\tilde{\mathcal{R}}_{L,R} = \mathcal{R}_{L,R}\left[i_{\vec{m}}(z,w) \to \tilde{i}_{\vec{m}}(z), j_{\vec{m}}(v,u) \to \tilde{j}_{\vec{m}}(v)\right], \tag{3.72}$$

and the regularized blocks are now given as

$$\begin{aligned}\tilde{i}_{\vec{m}} &= \frac{1}{\prod_{i=1}^{d(\vec{m})} m_i!}\int_0^\infty d\rho e^{-\rho}\prod_{i=1}^{d(\vec{m})}\left(\rho + n\alpha_{zz}^i\right)^{m_i}, \\ \tilde{j}_{\vec{m}} &= -\frac{\prod_{i=1}^{d(\vec{m})}(m_i-1)!}{2\pi i}\oint_{\Gamma_s} dp \frac{e^p(\gamma + \Gamma(0,p) + \ln p)}{\prod_{i=1}^{d(\vec{m})}(p + n\alpha_{vv}^i)^{m_i}}.\end{aligned} \tag{3.73}$$

The aforementioned regularization is conducted in the source paper (Grela and Guhr, 2016), we stress that it is an exact procedure in the following sense — although $\tilde{\mathcal{R}}_{L,R} \neq \mathcal{R}_{L,R}$, the spectral densities obtained by (3.65) agree exactly $\tilde{\rho}_N = \rho_N$. The final for-

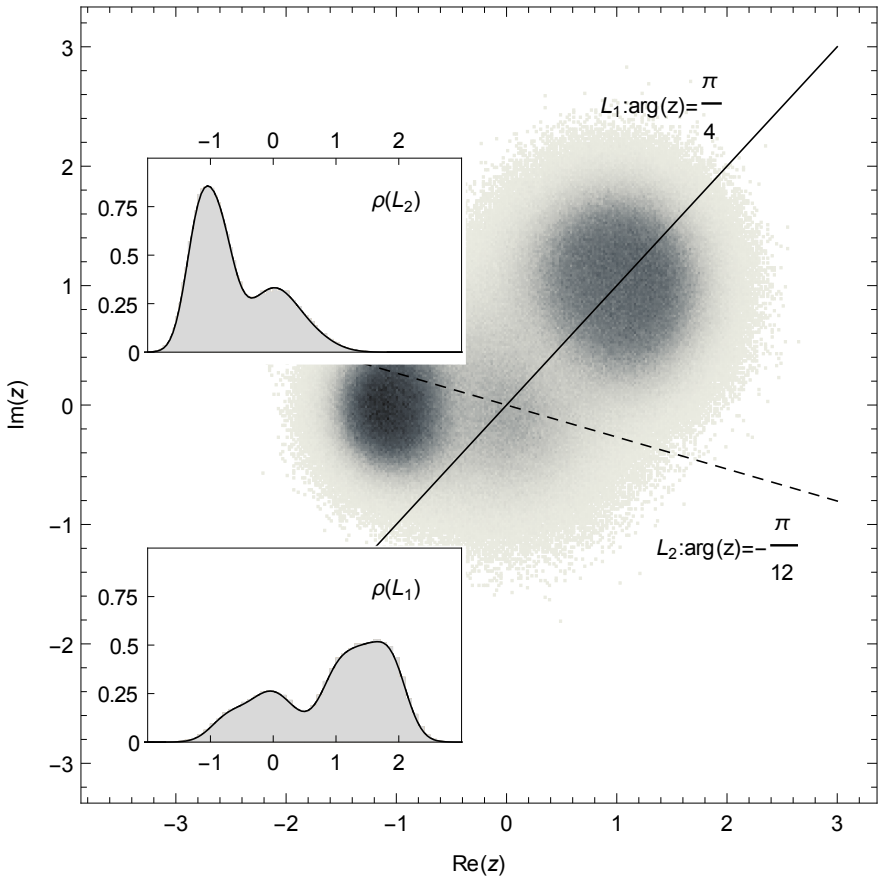

FIGURE 3.6: Numerical (background density plot) and analytic (insets) results for the normal S and diagonal L, R are presented. Spectral density formula of (3.74) is plotted in the insets along two straight lines L_1 and L_2 together with numerical simulations. The structural matrices are $S = \mathrm{diag}(-1, 0, 1 + i)$, $L = \mathrm{diag}(3/4, 1)$ and $R = \mathrm{diag}(1, 5/4, 1)$ with multiplicity vectors of $\vec{u} = (2, 1, 3)$, $\vec{v} = (2, 4)$ and $\vec{w} = (2, 1, 3)$.

mula for the spectral density in the case of a normal source S and non-trivial L, R then reads

$$\tilde{\rho}_N = -\frac{1}{N\pi}\frac{\partial}{\partial \bar{z}} \lim_{V \to Z} \frac{\partial}{\partial v} \tilde{\mathcal{R}}_{L,R}(z, v), \tag{3.74}$$

together with the definitions (3.68), (3.72) and (3.73). We demonstrate the utility of the analytical result (3.74) in Fig. 3.6 by comparing it with numerical simulations. Close connection to dynamical GinUE is clearly visible by inspection of the analogous Figures 3.3 and 3.4. Now however, the analytic formulae show perfect agreement with the numerics.

Non-normal rank one S and $L = R = 1$

A major reason to study models of the type (3.61) is the issue of spectral stability — how far do the eigenvalues of $S + Y$ spread around the eigenvalues of S for a small perturbation Y. This phenomenon was touched on in the Chapter 3.3.3 where the

initial inflation of the eigenvalues is traced back to the instability of the spectrum of a non-normal matrix X_0. The question of stability is especially interesting for finite rank sources S where extremal eigenvalues emerge from the sea of eigenvalues of Y (Tao, 2011; Tao, Vu, and Krishnapur, 2010; O'Rourke and Renfrew, 2014). We consider $L = R = 1$ and a rank one source of the form

$$S = \alpha \, |n\rangle \, \langle m| \, , \quad n \neq m, \tag{3.75}$$

for a complex parameter α and bra's (ket's) $\langle m|$ ($|n\rangle$) denoting the canonical matrix basis — the source matrix S has one non-zero element α placed on the off-diagonal. Computation of the ratio (3.63) in this case was conducted in the source paper (Grela

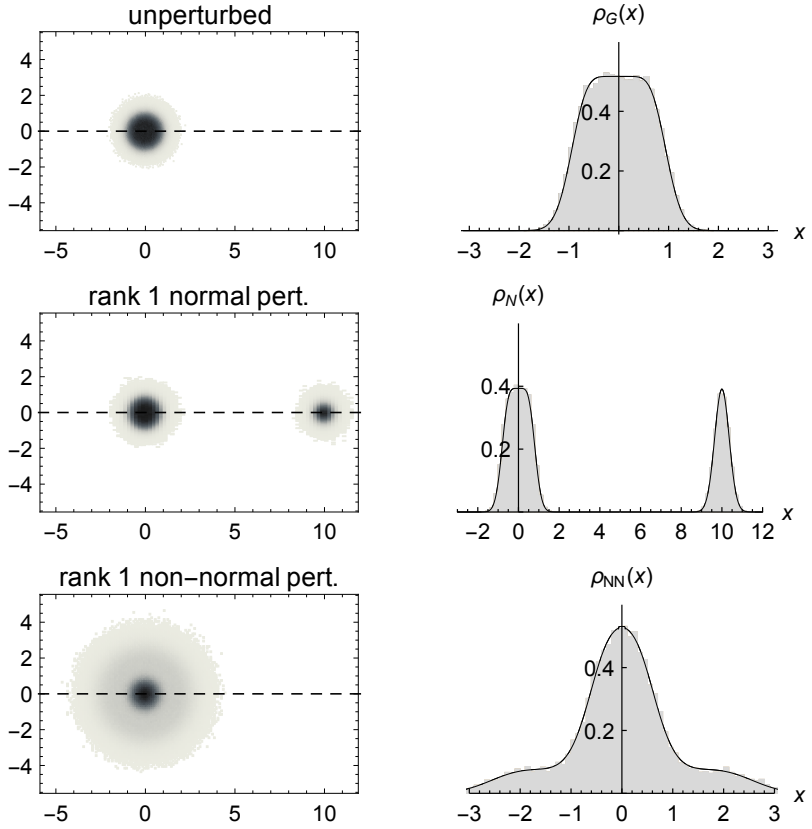

FIGURE 3.7: Left hand side: complex plane of eigenvalues, from top to bottom for: unperturbed $S = 0$ (Ginibre), normal perturbation $S = 10\,|1\rangle\,\langle 1|$ and non-normal perturbation $S = 10\,|2\rangle\,\langle 1|$. Right hand side: numerical simulations and analytical results for the spectral densities along the real axis line (dashed lines on the left hand side). Numerical simulations are for matrices of size $N = 4$, $\alpha = 10$, we set $n = N$.

and Guhr, 2016). Although the resulting formulae are lengthy, they are still composed of regularized fermionic building blocks $\tilde{i}_{k,l}$ ($k \neq 0, l = 1, 0, -1$) which we report on here:

$$\begin{aligned}
\tilde{i}_{k,0} &= \frac{(-1)^k}{n^{k+1}} \int_0^\infty d\rho e^{-\rho} (\rho + n|z|^2)^k, \\
\tilde{i}_{k,1} &= \tilde{i}_{k+2,0} - |\alpha|^2 (\tilde{i}_{k+1,0} + |z|^2 \tilde{i}_{k,0}), \\
\tilde{i}_{k,-1} &= \frac{(-1)^k k!}{(k_z^+ - k_z^-) n^k} \sum_{l=0}^{k} \frac{(n|z|^2)^l}{l!} \Big[U_{1,1+l-k}(nk_z^-) - U_{1,1+l-k}(nk_z^+) \Big],
\end{aligned} \tag{3.76}$$

where $k_x^{\pm} = \frac{1}{2}\left(|\alpha|^2 + 2|x|^2 \pm |\alpha|\sqrt{4|x|^2+|\alpha|^2}\right)$ and bosonic building blocks $\tilde{j}_{q,r}$ $(q + r \geq 1, r = 1, 2, 3)$ read:

$$\tilde{j}_{q,r} = -\frac{(-n)^{q+2r-1}}{2\pi i}\oint_\Gamma \frac{dpe^p \ln p}{(p+n|v|^2)^q (p+nk_v^-)^r (p+nk_v^+)^r}, \qquad q \geq 0, r = 1, 2, 3, \tag{3.77}$$

$$\tilde{j}_{-1,2} = \frac{1}{2}\left(\tilde{j}_{0,2_-} + \tilde{j}_{0,2_+} + |\alpha|^2 \tilde{j}_{0,2}\right), \quad \tilde{j}_{-1,3} = \frac{1}{2}\left(\tilde{j}_{0,3_-} + \tilde{j}_{0,3_+} + |\alpha|^2 \tilde{j}_{0,3}\right),$$
$$\tilde{j}_{-2,3} = \frac{1}{4}\left(\tilde{j}_{0,3_{--}} + 2\tilde{j}_{0,3_{+-}} + \tilde{j}_{0,3_{++}} + |\alpha|^4 \tilde{j}_{0,3} + 2|\alpha|^2(\tilde{j}_{0,3_+} + \tilde{j}_{0,3_-})\right).$$

The contour Γ encircles counter-clockwise both $-n|v|^2$ and $-nk_v^{\pm}$ and the subscripts $\pm$ indicate that the underlying multiplicity vector $\vec{x} = (q, r-1, r)$ is applied with decrement to the source at $nk_v^{\pm}$.

Analytic formulae for the spectral density are presented in Fig. 3.7. To facilitate a comparison, we juxtapose it with the analogous results for the case of a rank one normal source S and for the Ginibre case. A non-normal source S (third row in Fig. 3.7) does not produce, on average, outlier eigenvalues in the spectrum, in contrast to a normal source S (second row in Fig. 3.7) where we find an island around $\alpha = 10$. Instead, in the non-normal case we observe an overall blow-up of the spectral bulk. For comparison, the first row in Fig. 3.7 is devoted to the case of a vanishing source, $S = 0$.

Spectrum of M^{-1}

As a last application we discuss on how to obtain somewhat gratuitously the spectrum of M^{-1} from the results for the spectrum of M. We consider a normal source S and set $L = R = 1$. To this end, we define a generating function $\mathcal{R}_{-1}$ for the inverse as

$$\mathcal{R}_{-1}(Z, V) \equiv \left\langle \frac{\det(Z - \mathcal{M}_{-1})}{\det(V - \mathcal{M}_{-1})} \right\rangle_P = \frac{\det Z}{\det V}\mathcal{R}_{1,1}\left(Z', V'\right), \tag{3.78}$$

and relate it to the generating function (3.63) considered previously. The matrices $\mathcal{M}_{-1} = \begin{pmatrix} 0 & M^{-1} \\ M^{\dagger,-1} & 0 \end{pmatrix}$ and Z', V' are rearranged versions of the inverse matrices Z^{-1}, V^{-1} of (3.64),

$$X' = \begin{pmatrix} (X^{-1})_{22} & (X^{-1})_{21} \\ (X^{-1})_{12} & (X^{-1})_{11} \end{pmatrix}, \quad X = Z, V. \tag{3.79}$$

The calculation described in Chapter 3.5.2 is repeated with the replacements $w \to -wG_{zw}, z \to \bar{z}G_{zw}, u \to -uG_{vu}$ and $v \to \bar{v}G_{vu}$ with $G_{xy} = (|x|^2 + |y|^2)^{-1}$. After conducting the regularization procedure we eventually find that the source matrix of (3.67) gets modified

$$\alpha_{xy} \to (\alpha^{-1})_{xy} = \alpha_{x^{-1}y^{-1}} = (\bar{x}^{-1}\mathbf{1}_N - S^\dagger)(y^{-1}\mathbf{1}_N - S).$$

The regularized ratio ford the spectrum of $(S + X)^{-1}$ reads

$$\tilde{\mathcal{R}}_{-1} = \left(\frac{|z|^2}{|v|^2}\right)^{|\vec{n}|} \tilde{\mathcal{R}}_{1,1}\left[\alpha_{xy} \to (\alpha^{-1})_{xy}\right], \tag{3.80}$$

where the generating function $\tilde{\mathcal{R}}_{1,1}$ is that of (3.72) and the constituent fermionic and bosonic blocks (3.73) are affected accordingly. In particular, we calculate the spectral density for an inverse matrix X^{-1} as

$$\rho_N^{\mathrm{G}^{-1}} = \frac{n e^{-\frac{n}{|z|^2}}}{N\pi|z|^4} \sum_{k=0}^{N-1} \frac{1}{(k)!} \left(\frac{n}{|z|^2}\right)^k , \tag{3.81}$$

which was also found in a recent work on the product of matrices (Adhikari et al., 2016).

Chapter 4

Diffusion method

Throughout this dissertation, introducing dynamical matrices proved fruitful in obtaining new results, gaining insight into the structure of random matrix models and providing a unifying hydrodynamical framework. At the backbone of most of the results, a similar approach was used — the basic formulae of (2.28), (2.29) and (3.21) follow a common scheme which we describe in this concluding part. To make this approach concrete, we have formalized it in the paper (Grela, 2016) under the name of diffusion method. The name follows from the fact that the matrix dynamics in question are always diffusive in character as they are described by the Smoluchowski–Fokker–Planck equation.

4.1 Descripton of the method

The objects of interest are the ratios and products of characteristic polynomials denoted as $D(Z, M)$ with Z denoting a set of arguments of characteristic polynomials and M is the matrix in question. We are interested in finding formulae for the average

$$\overline{D_t}(Z) \equiv \langle D(Z, M_t) \rangle \,, \tag{4.1}$$

where the averaged is computed over the dynamical matrices M_t. Matrix M_t is drawn from a dynamical GβE of Tab. 2.1, dynamical GinβE of Tab. 3.1 or other ensemble defined on the level of matrix entries. We repeat the SFP equation of (2.16) for clarity:

$$\partial_t P_t(M|M_0) = \Delta_M P_t(M|M_0). \tag{4.2}$$

In the first step, we extend the object $D(Z, M) \to \mathbf{D}(Z, M; \Lambda)$ by introducing additional parameter variables Λ such that $\lim_{\Lambda \to \Lambda_0} \mathbf{D}(Z, M; \Lambda) = D(Z, M)$. An algorithm for constructing Λ's will be characterised subsequently. The purpose of this extension is however clear — we search for a dual diffusive equation for the averaged deformed quantity $\overline{\mathbf{D}_t}(Z; \Lambda) \equiv \langle \mathbf{D}(Z, M_t; \Lambda) \rangle$ in the Λ-parameter space.

In order to find it, we consider a time derivative of $\overline{\mathbf{D}_t} = \int d[M] P_t(M|M_0) \mathbf{D}(Z, M; \Lambda)$:

$$\partial_t \overline{\mathbf{D}_t} = \int dM \Delta_M P_t(M|M_0) \mathbf{D}(Z, M; \Lambda) = \int d[M] P_t(M|M_0) \Delta_M \mathbf{D}(Z, M; \Lambda), \tag{4.3}$$

where we used the equation (4.2) and integrated by parts to move the differential operator to $\mathbf{D}$. Note that for Gaussian Δ_M (i.e. containing only second derivatives) integration by parts is tractable and does not produce any boundary terms for well-behaving functions P_t and $\mathbf{D}$. The remaining task is to find $\mathbf{\Delta}_\Lambda$ such that the condition

$$\Delta_M \mathbf{D}(Z, M; \Lambda) = \mathbf{\Delta}_\Lambda \mathbf{D}(Z, M; \Lambda) \tag{4.4}$$

is satisfied so that we calculate

$$\partial_t \overline{\mathbf{D}_t} = \int d[M] P_t(M|M_0) \Delta_M \mathbf{D}(Z, M; \Lambda) = \mathbf{\Delta}_\Lambda \int d[M] P_t(M|M_0) \mathbf{D}(Z, M; \Lambda) = \\ = \mathbf{\Delta}_\Lambda \overline{\mathbf{D}_t}(Z; \Lambda),$$

and write the dual diffusive equation as

$$\partial_t \overline{\mathbf{D}_t}(Z; \Lambda) = \mathbf{\Delta}_\Lambda \overline{\mathbf{D}_t}(Z; \Lambda). \tag{4.5}$$

As can be seen from the condition (4.4), Gaussian Laplace operators acting on the M manifold are transformed into Gaussian Laplace operators on the Λ space but, at the same time, we observe a decrease in the number of variables. This fact enables us to effectively solve an initial value problem of (4.5) with a heat kernel K_t:

$$\overline{\mathbf{D}_t}(Z; \Lambda) = K_t(\Lambda, \Lambda') \circ \overline{\mathbf{D}_{t=0}}(Z; \Lambda'), \tag{4.6}$$

where the symbol "$\circ$" denotes a convolution operator and K_t is defined by $(\partial_t - \mathbf{\Delta}_\Lambda) K_t = 0$, $\lim_{t \to 0} K_t(\Lambda, \Lambda') = \delta(\Lambda - \Lambda')$. As a last step, the non-deformed average reads

$$\overline{D_t}(Z) = \lim_{\Lambda \to \Lambda_0} K_t(\Lambda, \Lambda') \circ \overline{\mathbf{D}_{t=0}}(Z; \Lambda'). \tag{4.7}$$

Concrete forms of K_t are known once we specify the problem at hand.

Constructing deformations Λ so that the condition (4.4) is satisfied is crucial. A way to do this operationally is to open the determinants with the use of Grassmann or complex representations and evaluate the l.h.s. of (4.4). Afterwards, introduce additional parameters into the representation such that derivatives with respect to Λ give the r.h.s. of (4.4). This procedure and its usefulness was discussed in full detail in the work (Grela, 2016). In the simplest cases we have already implemented this method — in the case of dynamical GUE discussed in Chapter 2.3 no additional parameters Λ (besides Z) were needed to obtain either (2.28) or (2.29). In the dynamical GinUE discussed in Chapter 3.1, the Λ space consists of the w variable as in (3.18). The deformation in this case is simply:

$$\det(z - X) \det(\bar{z} - X^\dagger) \quad \longrightarrow \quad \det\left[(z - X)(\bar{z} - X^\dagger) + |w|^2\right]. \tag{4.8}$$

4.2 Some examples

This section is devoted to several examples and serves as a tour-de-force showing the framework at work to calculate new results and compare to known ones.

First example is devoted to the dynamical GUE. We show the applicability of our method to the averaged ratio of determinants, we obtain an integral representation for any initial matrix H_0 reproducing the formula of (2.47) and completing the discussion in Chapter 2.3.1.

Second example elucidates on a certain duality-type formula for dynamical GinUE, a result which continues the successful programme of dualities obtained in both the GUE (Forrester and Witte, 2001; Desrosiers, 2009) and the GinUE (Akemann and Vernizzi, 2003).

Last example is a toy-model used to study the crossover between the $\beta = 1$ and $\beta = 2$ dynamical Ginibre Ensembles. We arrive at the large N formula of the real-axis bump which develops while we vary the crossover parameter.

We describe two additional cases considered in (Grela, 2016) and relegate interested reader to the paper for the details. First one serves is the application of the method to the multiplication of several independent dynamical GinUE matrices which attracted a lot of attention recently (Akemann and Burda, 2012; Ipsen, 2013; Kuijlaars and Zhang, 2014). Second is a calculation of the AECP in a dynamical GinUE model with variance structure, considered in (Forrester and Rains, 2009).

4.2.1 Ratio of determinants in dynamical GUE

In this example we consider the ratio of characteristic polynomials and set $M \equiv H$ throughout:

$$D(z, w, H) = \frac{\det(z - H)}{\det(w - H)}, \tag{4.9}$$

and let H be a dynamical GUE matrix. We read off the form of the SFP equation for the jPDF from the second row of Tab. 2.1:

$$\partial_\tau P_\tau(H|H_0) = \frac{1}{2N} \left(\sum_{k=1}^{N} \partial_{x_{kk}}^2 + \frac{1}{2} \sum_{i>j=1}^{N} \left(\partial_{x_{ij}}^2 + \partial_{y_{ij}}^2 \right) \right) P_\tau(H|H_0),$$

where we have rescaled the time variable $Nt = \tau$. We open the determinants by using the Grassmann/complex representations

$$D(z, w, H) \sim \int d[\eta, \alpha] \exp \left[- \begin{pmatrix} \bar{\alpha} & \bar{\eta} \end{pmatrix} \begin{pmatrix} w - H & 0 \\ 0 & z - H \end{pmatrix} \begin{pmatrix} \alpha \\ \eta \end{pmatrix} \right]. \tag{4.10}$$

where the exact proportionality constant is not essential to what follows and the $d[\eta, \alpha]$ measure is defined accordingly. The deformation Λ consists of two Grassmann variables p, q which replace the off-diagonal zeroes in the formula (4.10) and thus the deformed object $\mathbf{D}(z, w, M; \Lambda)$ reads

$$\mathbf{D}(z, w, H; q, p) \sim \int d[\eta, \alpha] \exp \left[- \begin{pmatrix} \bar{\alpha} & \bar{\eta} \end{pmatrix} \begin{pmatrix} w - H & q \\ p & z - H \end{pmatrix} \begin{pmatrix} \alpha \\ \eta \end{pmatrix} \right], \tag{4.11}$$

which is discussed in detail in the paper (Grela, 2016). By direct calculation we find that the operator Δ_Λ fulfilling the condition (4.4) is equal to $\mathbf{\Delta}_\Lambda = \frac{1}{2} (\partial_{ww} - \partial_{zz} - 2\partial_p \partial_q)$. The dual diffusion-like equation (4.5) therefore reads

$$\partial_\tau \overline{\mathbf{D}_\tau}(z, w; p, q) = \frac{1}{2N} (\partial_{ww} - \partial_{zz} - 2\partial_p \partial_q) \, \overline{\mathbf{D}_\tau}(z, w; p, q). \tag{4.12}$$

We comment on two features of (4.12) — in the z direction it has a negative diffusivity constant and the diffusion happens also in the p, q Grassmann "directions". On a technical level we deal with it by Wick-rotating the $z \to iz$ variable. Secondly, the p and q variables are Grassmann numbers and thus we utilize the "flatness" property

$q^2 = 0, p^2 = 0$ to expand the equation (4.11)

$$\mathbf{D}(z,w,H;p,q) = \mathbf{D}^{(1)}(z,w,H) + p\mathbf{D}^{(2)}(z,w,H) + q\mathbf{D}^{(3)}(z,w,H) + qp\mathbf{D}^{(4)}(z,w,H). \tag{4.13}$$

In terms of the determinants:

$$\mathbf{D}^{(1)}(z,w,H) = \frac{\det(z-H)}{\det(w-H)}, \tag{4.14}$$

$$\mathbf{D}^{(4)}(z,w,H) = \frac{\det(z-H)}{\det(w-H)}\mathrm{Tr}\frac{1}{(z-H)(w-H)}. \tag{4.15}$$

and with two other components vanishing $\mathbf{D}^{(2)}(z,w,H) = \mathbf{D}^{(3)}(z,w,H) = 0$. Following the Grassmann expansion of (4.13), the equation (4.12) for the averaged object $\overline{\mathbf{D}_\tau}$ is equivalent to a system of 4 equations for each component $\overline{\mathbf{D}_\tau}^{(i)}$:

$$\partial_\tau \overline{\mathbf{D}_\tau}^{(1)} = \frac{1}{2N}\left(\partial_{ww} - \partial_{zz}\right)\overline{\mathbf{D}_\tau}^{(1)} - \frac{1}{N}\overline{\mathbf{D}_\tau}^{(4)}, \tag{4.16}$$

$$\partial_\tau \overline{\mathbf{D}_\tau}^{(2)} = \frac{1}{2N}\left(\partial_{ww} - \partial_{zz}\right)\overline{\mathbf{D}_\tau}^{(2)}, \tag{4.17}$$

$$\partial_\tau \overline{\mathbf{D}_\tau}^{(3)} = \frac{1}{2N}\left(\partial_{ww} - \partial_{zz}\right)\overline{\mathbf{D}_\tau}^{(3)}, \tag{4.18}$$

$$\partial_\tau \overline{\mathbf{D}_\tau}^{(4)} = \frac{1}{2N}\left(\partial_{ww} - \partial_{zz}\right)\overline{\mathbf{D}_\tau}^{(4)}. \tag{4.19}$$

where we have suppressed the arguments of $\overline{\mathbf{D}_\tau}^{(i)} = \overline{\mathbf{D}_\tau}^{(i)}(z,w)$. To find the solution we recall that the interesting observable is the ratio of (4.9) which is retrieved by taking the non-deformed limit $p, q \to 0$ i.e. $\overline{D_\tau} = \lim_{p,q\to 0}\overline{\mathbf{D}_\tau} = \overline{\mathbf{D}_\tau}^{(1)}$. Therefore, only equations (4.16) and (4.19) contain relevant information. To solve them we form a heat kernel of the Laplace operator $\frac{1}{2N}(\partial_{ww} - \partial_{zz})$:

$$K_\tau(z,w;y,v) = \frac{N}{2\pi\tau}\exp\left(-\frac{N}{2\tau}(v-w)^2 - \frac{N}{2\tau}(y-iz)^2\right), \tag{4.20}$$

which turns out to be a combined heat kernel of the evolution equation for both the ACP (2.30) and the AICP (2.31). This in turn renders the following discussion largely parallel — the y, z arguments were Wick rotated whereas the v, w variables define by two different functions depending on the sign of $\mathrm{Im}v$ as in (2.32). The solution to the equation (4.19) reads

$$\overline{\mathbf{D}_\tau}^{(4),\pm}(z,w) = \int_{-\infty}^{\infty} dy \int_{\Gamma_\pm} dv K_\tau(z,w;y,v)\mathbf{D}_0^{(4)}(-iy,v,H_0) \equiv \left(K_\tau \overset{\pm}{\circ} \mathbf{D}_0^{(4)}\right)(z,w),$$

with H_0 denoting the initial matrix and the superscript $\pm$ denoting two different solutions corresponding to different contours $\Gamma_\pm$ encircling the poles of $\mathbf{D}_0^{(4)}$ from above $(+)$ or below $(-)$. We refer to Chapter 2.3.1 for a thorough analysis of these cases. With this notation however, the solution to the inhomogeneous heat equation (4.16) is simply equal to

$$\overline{\mathbf{D}_\tau}^{(1),\pm}(z,w) = \left(K_\tau \overset{\pm}{\circ} \left(\mathbf{D}_0^{(1)} - \frac{\tau}{N}\mathbf{D}_0^{(4)}\right)\right)(z,w), \tag{4.21}$$

which, by plugging the kernel (4.20) and the initial condition (4.15), is explicitly given as

$$\overline{\mathbf{D}_\tau}^{(1),\pm}(z,w) = \frac{N}{2\pi\tau}\int_{-\infty}^{\infty} dy \int_{\Gamma_\pm} dv \exp\left(-\frac{N}{2\tau}(v-w)^2 - \frac{N}{2\tau}(y-iz)^2\right) \times \\ \times \frac{\det(-iy-H_0)}{\det(v-H_0)}\left(1-\frac{\tau}{N}\mathrm{Tr}\frac{1}{(-iy-H_0)(v-H_0)}\right). \quad (4.22)$$

With no loss of generality we consider diagonal H_0 of the form of (2.41) with d eigenvalues a_i ($i = 1...d$) of degeneracies n_i ($i = 1...d$). To arrive at a unique formula for $\mathrm{Im}w = 0$ we define the object $\overline{\mathbf{D}_\tau}^{(1)} = \frac{1}{2\pi i}\left(\overline{\mathbf{D}_\tau}^{(1),-} - \overline{\mathbf{D}_\tau}^{(1),+}\right)$ and use the definitions of multiple orthogonal polynomials of (2.42) and (2.43) to arrive at:

$$\overline{\mathbf{D}_\tau}^{(1)}(z,w) = \overline{D_\tau}(z,w) = \Pi_{\vec{n}}(z)\Theta_{\vec{n}}(w) - \frac{\tau}{N}\sum_{i=1}^{N}\Pi_{\vec{n}-\vec{e}_i}(z)\Theta_{\vec{n}+\vec{e}_i}(w), \quad (4.23)$$

where $\vec{n} = (n_1...n_d)$ and $\vec{e}_i = (0,0,...,1,...,0)$ is the d–dimensional unit vector with unity at the i–th position. To complete the picture we cite a Christoffel–Darboux type formula of (Daems and Kuijlaars, 2004):

$$(z-w)\sum_{i=0}^{N-1}\Pi_i(z)\Theta_i(w) = \Pi_{\vec{n}}(z)\Theta_{\vec{n}}(w) - \frac{\tau}{N}\sum_{i=1}^{d}\Pi_{\vec{n}-\vec{e}_i}(z)\Theta_{\vec{n}+\vec{e}_i}(w), \quad (4.24)$$

where the polynomials Π_i and Θ_i are defined in (2.45). We readily notice that the l.h.s. of (4.24) is the kernel of (2.46) times $(z-w)$ whereas the r.h.s. is exactly the calculated ratio of determinants (4.23). Therefore, we recreate the result of (Desrosiers and Forrester, 2008) that the kernel is proportional to the averaged ratio of determinants.

4.2.2 Dualities of dynamical GinUE model

We define $M \equiv X$ to be a dynamical matrix of GinUE type, from the second row of Tab. 3.1 we read off the SFP equation as

$$\partial_\tau P_\tau(X|X_0) = \frac{1}{4N}\sum_{i,j=1}^{N}\left(\partial^2_{x_{ij}} + \partial^2_{y_{ij}}\right)P_\tau(X|X_0),$$

where again we set $Nt = \tau$. We aim at calculating an averaged product of k extended characteristic polynomials

$$D^{(k)}(\mathcal{Z},X) = \det\left[\prod_{i=1}^{k}(z_i - X)(\bar{z}_i - X^\dagger)\right]. \quad (4.25)$$

Deformation is in turn a $2kN$ block matrix of the form

$$\mathbf{D}^{(k)}(\mathcal{Z},X;A) = \det\begin{pmatrix}\mathcal{Z}\otimes\mathbf{1}_N - \mathbf{1}_k\otimes X & -A^\dagger\otimes\mathbf{1}_N \\ A\otimes\mathbf{1}_N & \mathcal{Z}^\dagger\otimes\mathbf{1}_N - \mathbf{1}_k\otimes X^\dagger\end{pmatrix}, \quad (4.26)$$

where $\mathcal{Z} = \mathrm{diag}(z_1,...z_k)$, $\mathbf{1}_n$ is a $n\times n$ dimensional unit matrix and A is a complex $k\times k$ matrix representing the Λ parameter space. We baptise the object $\mathbf{D}^{(k)}$ as the k–Averaged Extended Characteristic Polynomial (or k–AECP) since it is a direct generalization of (3.18) where a particular case of $k = 1$ and the deformation parameter

$A = w$ was considered. The dual equation in the Λ parameter space is simply given as

$$\mathbf{\Delta}_\Lambda = \frac{1}{4}\sum_{i,j=1}^{k}\left(\partial^2_{a_{ij}} + \partial^2_{b_{ij}}\right) \equiv \mathrm{Tr}\partial_{AA^\dagger}, \tag{4.27}$$

where $A_{kl} = a_{kl} + ib_{kl}$. For details of arriving at (4.27) we again relegate to the source paper (Grela, 2016). The equation for an average $\overline{\mathbf{D}_\tau}^{(k)}(\mathcal{Z}; A)$ reads:

$$\partial_\tau \overline{\mathbf{D}_\tau}^{(k)}(\mathcal{Z}; A) = \frac{1}{N}\mathrm{Tr}\partial_{AA^\dagger}\overline{\mathbf{D}_\tau}^{(k)}(\mathcal{Z}; A), \tag{4.28}$$

where we observe a dimensional reduction $N \times N \to k \times k$ which is characteristic in a matrix-type duality formulae. To arrive at a concrete equation, we write down the solution of (4.28) after taking the non-deformed limit $A \to 0$:

$$\overline{D_\tau}^{(k)}(\mathcal{Z}) = \left(\frac{N}{\pi\tau}\right)^{k^2}\int_{k\times k} d[B]e^{-\frac{N}{\tau}\mathrm{Tr}B^\dagger B}\mathbf{D}^{(k)}(\mathcal{Z}, X_0; B), \tag{4.29}$$

where X_0 is the initial matrix. On the other hand, from the definition we write down the average $\overline{D_\tau}^{(k)}(\mathcal{Z})$ as:

$$\overline{D_\tau}^{(k)}(\mathcal{Z}) = \left(\frac{N}{\pi\tau}\right)^{N^2}\int_{N\times N} d[X]e^{-\frac{N}{\tau}\mathrm{Tr}X^\dagger X}\mathbf{D}^{(k)}(\mathcal{Z}, X + X_0; 0). \tag{4.30}$$

where we changed $X \to X + X_0$ and used the identity $D^{(k)}(\mathcal{Z}, M) = \mathbf{D}^{(k)}(\mathcal{Z}, M + M_0; 0)$. We write down the duality formula by equating (4.29) and (4.30):

$$\begin{aligned}&\left(\frac{N}{\pi\tau}\right)^{N^2}\int_{N\times N} d[X]e^{-\frac{N}{\tau}\mathrm{Tr}X^\dagger X}\mathbf{D}^{(k)}(\mathcal{Z}, X + X_0; 0) = \\ &= \left(\frac{N}{\pi\tau}\right)^{k^2}\int_{k\times k} d[B]e^{-\frac{N}{\tau}\mathrm{Tr}B^\dagger B}\mathbf{D}^{(k)}(\mathcal{Z}, X_0; B),\end{aligned} \tag{4.31}$$

with the integrand $\mathbf{D}^{(k)}$ defined in (4.26).

This new result is an extension of a similar formula for $X_0 = 0$ obtained in (Akemann and Vernizzi, 2003).

4.2.3 GinOE/GinUE crossover model

The last example is a crossover model between the real ($\beta = 1$) and complex ($\beta = 2$) Ginibre Ensembles. A matrix drawn from a GinOE$_{\beta=1}$ have either real or complex conjugated pairs of eigenvalues whereas GinUE$_{\beta=2}$ is not constrained by such condition — its eigenvalues spread evenly over the complex plane. To study this transition, we combine the dynamical models of first two rows of Tab. 3.1 in the following SFP equation:

$$\partial_\tau P_\tau(X|X_0) = \frac{1}{4N}\sum_{i,j=1}^{N}\left(\partial^2_{x_{ij}} + \alpha^2\partial^2_{y_{ij}}\right)P_\tau(X|X_0).$$

which forms an $N \times N$ matrix $X_{kl} = x_{kl} + iy_{kl}$. The model contains a crossover parameter α which varies between 0 ($\beta = 1$) and 1 ($\beta = 2$). We investigate the condensation of eigenvalues on the real line as we take the limit $\alpha \to 0$. To this end we study the

AECP of (3.18):

$$D(z,X) = \det(z-X)\det(\bar{z}-X^\dagger), \tag{4.32}$$

and the deformed object reads:

$$\mathbf{D}(z,X;w) = \det\begin{pmatrix} z-X & -\bar{w} \\ w & \bar{z}-X^\dagger \end{pmatrix}. \tag{4.33}$$

We report on a dual diffusion equation

$$\partial_\tau \overline{\mathbf{D}_\tau}(z;w) = \frac{1+\alpha^2}{2N}\partial_{w\bar{w}}\overline{\mathbf{D}_\tau}(z;w). \tag{4.34}$$

which reduces to (3.21) as $\alpha \to 1$. The solution, after taking the non-deformed $w \to 0$ limit, reads

$$\overline{D_\tau}(z) = \frac{2N}{\tau}\int_0^\infty dr r e^{-\frac{2N}{\tau(1+\alpha^2)}r^2}\mathbf{D}(z,X_0;r), \tag{4.35}$$

which is valid for any initial matrix X_0. For vanishing $X_0 \to 0$, the formula (4.35) agrees with the results for both Gin($\beta = 1,2$)E (Akemann, Phillips, and Sommers, 2009; Burda et al., 2015).

We now turn to a microscopic crossover region of $\alpha \to 0$ and $\mathrm{Im}z \to 0$ where a precursor of the real eigenvalue comb of the GinOE is visible. We set a microscopic scaling near the real axis $-N^{1/4}iz = \eta$ and the crossover parameter near zero $a = N^{1/4}\alpha$ which yields an asymptotic formula:

$$\overline{D_\tau}(i\eta N^{-1/4}; \alpha = aN^{-1/4}) \sim c_N e^{-a^4/2} e^{-\frac{2a^2\eta^2}{\tau}} \mathrm{erfc}\left(\frac{\sqrt{2}\eta^2}{\tau} - \frac{a^2}{\sqrt{2}}\right),$$

where the N dependent part c_N was skipped for clarity. This result clearly shows an error function type bump near $\eta = 0$ which we interpret as the discussed precursor of an emerging bulk of real eigenvalues.

Chapter 5

Conclusions and outlook

In this dissertation we have studied dynamical matrices of Hermitian and non-Hermitian type. Since we are particularly interested in the dynamics of the eigenvalues and eigenvectors, main distinction of these two groups is that the eigenvalues occupy one-dimensional domains in the former and two-dimensional domains in the latter. Despite this, they share a hydrodynamical description — in the Hermitian models, the eigenvalues form a fluid-like system whereas in the non-Hermitian case, an amalgam of eigenvalues and eigenvectors form the hydrodynamics.

Different Hermitian models were considered to corroborate the robustness of this hydrodynamical idea. The resulting Burgers' equations are intimately linked to the one-dimensional gas of eigenvalues as we found throughout this thesis. We pursued a particular path through the averaged characteristic polynomial to arrive at this fluid-like picture. We have used the collective variables technique as another way of introducing the hydrodynamics which additionally made possible the asymptotic results for the angular integrals of the Harish-Chandra/Itzykson–Zuber and Berezin–Karpelevich type.

In the non-Hermitian dynamical matrices, a fluid-like description is available — the difference is that in this case the fluid is described by the eigenvalues weighted with eigenvectors. By noticing this extension we again recover the hydrodynamics although in a slightly different degrees of freedom. Identifying and interpreting these new variables is crucial in understanding the non-Hermitian random matrices. During this endeavour we derive non-Hermitian Burgers' equations, we find macroscopic formulas for the spectral densities and the eigenvector correlators and we identify the relevance of non-normal initial conditions.

The main results of this thesis was found by an approach which was formalized under the name of the diffusion method. It is a computational tool of finding the averages of determinants. It is simply describable as a two-step procedure which assumes diffusive dynamics of matrices and look for similar diffusive equations for the observables but now in a dual space of auxiliary variables.

To conclude, we discuss possible further developments. As we have seen in preliminary studies of dynamical Ginibre Unitary Ensembles in Chapter 3.1.2, a concrete form of non-Hermitian SFP equation remains to be discovered. It would be also extremely interesting to look for higher order (i.e. two-point and further) eigenvector correlators of (3.13) which may reveal a hidden structure similar to the one present in the higher-order eigenvalue correlation functions. Another possible direction is to complete the dynamical picture by looking at non-equal time correlators. The topic of non-Hermitian random matrices remains both interesting and not yet fully discovered.

Appendix A

Dyson's perturbative derivation

We consider the eigenvalue equation of (2.18):

$$H_{t'} \left|\psi_i(t')\right\rangle = \lambda_i(t') \left|\psi_i(t')\right\rangle , \tag{A.1}$$

and use the definition of (2.13) to set $H_{t'} = H_t + \delta H$. Likewise, the eigenvectors and eigenvalues read

$$\lambda_i(t') = \lambda_i(t) + \delta\lambda_i, \qquad \left|\psi_i(t')\right\rangle = \left|\psi_i(t)\right\rangle + \delta \left|\psi_i\right\rangle .$$

A perturbative approach is possible upon identifying the small parameter ϵ. To this end, we set $H_t \to H, \delta H \to \epsilon\delta H$ and identify

$$\left|\psi_i(t)\right\rangle \to \left|\psi_i^{(0)}\right\rangle , \qquad \delta \left|\psi_i\right\rangle \to \epsilon \left|\psi_i^{(1)}\right\rangle + \epsilon^2 \left|\psi_i^{(2)}\right\rangle + \cdots , \tag{A.2}$$

$$\lambda_i(t) \to \lambda_i^{(0)}, \qquad \delta\lambda_i \to \epsilon\lambda_i^{(1)} + \epsilon^2\lambda_i^{(2)} + \cdots . \tag{A.3}$$

In this context, introducing ϵ is just a way of systematically indexing the order of corrections and we set it to unity in the end. Transformed equation (A.1) now reads:

$$\begin{aligned}(H + \epsilon\delta H) \left(\left|\psi_i^{(0)}\right\rangle + \epsilon \left|\psi_i^{(1)}\right\rangle + \epsilon^2 \left|\psi_i^{(2)}\right\rangle + \cdots\right) = \\ = \left(\lambda_i^{(0)} + \epsilon\lambda_i^{(1)} + \epsilon^2\lambda_i^{(2)} + \cdots\right) \left(\left|\psi_i^{(0)}\right\rangle + \epsilon \left|\psi_i^{(1)}\right\rangle + \epsilon^2 \left|\psi_i^{(2)}\right\rangle + \cdots\right) ,\end{aligned} \tag{A.4}$$

Zeroth order. We collect the terms of (A.4) proportional to ϵ^0:

$$H \left|\psi_i^{(0)}\right\rangle = \lambda_i^{(0)} \left|\psi_i^{(0)}\right\rangle , \tag{A.5}$$

which is just the unperturbed equation.

First order. Terms in (A.4) proportional to ϵ read:

$$\delta H \left|\psi_i^{(0)}\right\rangle + H \left|\psi_i^{(1)}\right\rangle = \lambda_i^{(0)} \left|\psi_i^{(1)}\right\rangle + \lambda_i^{(1)} \left|\psi_i^{(0)}\right\rangle , \tag{A.6}$$

which, by acting with $\langle\psi_j^{(0)}|$, is given as:

$$\delta H_{ji} + \lambda_j^{(0)} \left\langle\psi_j^{(0)}|\psi_i^{(1)}\right\rangle = \lambda_i^{(0)} \left\langle\psi_j^{(0)}|\psi_i^{(1)}\right\rangle + \lambda_i^1\delta_{ij}, \tag{A.7}$$

where $\delta H_{ij} = \langle\psi_i^{(0)}| \delta H \left|\psi_j^{(0)}\right\rangle$. For $i = j$ we obtain the first eigenvalue correction:

$$\lambda_i^{(1)} = \delta H_{ii}, \tag{A.8}$$

whereas for $i \neq j$:

$$\langle \psi_j^{(0)} | \psi_i^{(1)} \rangle = \frac{\delta H_{ji}}{\lambda_i^{(0)} - \lambda_j^{(0)}}, \tag{A.9}$$

we obtain the first correction to the eigenvector:

$$|\psi_i^{(1)}\rangle = \langle \psi_i^{(0)} | \psi_i^{(1)} \rangle \, |\psi_i^{(0)}\rangle + \sum_{k(\neq i)} \frac{\delta H_{ki}}{\lambda_i^{(0)} - \lambda_k^{(0)}} \, |\psi_k^{(0)}\rangle \, . \tag{A.10}$$

We encounter the amplitude $\langle \psi_i^{(0)} | \psi_i^{(1)} \rangle$ which we specify by imposing the orthogonality condition of corrected eigenvectors. Specifically, orthogonality relation

$$(\langle \psi_i | + \delta \, \langle \psi_i |)(|\psi_j\rangle + \delta \, |\psi_j\rangle) = \delta_{ij}. \tag{A.11}$$

To the first order of ϵ it is equal to:

$$\langle \psi_i^{(1)} | \psi_j^{(0)} \rangle + \langle \psi_i^{(0)} | \psi_j^{(1)} \rangle = 0. \tag{A.12}$$

For $i \neq j$ it is always satisfied since, by plugging in (A.9) we obtain an identity. For $i = j$ we have

$$\mathrm{Re} \, \langle \psi_i^{(1)} | \psi_i^{(0)} \rangle = 0. \tag{A.13}$$

Therefore, the real part of $\langle \psi_i^{(1)} | \psi_i^{(0)} \rangle$ should vanish if corrected eigenvectors are to be orthonormal. Moreover, imaginary part can also be put to zero since the correction $|\psi_i^{(1)}\rangle$ proportional to $|\psi_i^{(0)}\rangle$ can be ultimately absorbed by the unperturbed eigenvector and the freedom of eigenvector normalization. Therefore, finally the first order eigenvector correction reads

$$|\psi_i^{(1)}\rangle = \sum_{k(\neq i)} \frac{\delta H_{ki}}{\lambda_i^{(0)} - \lambda_k^{(0)}} \, |\psi_k^{(0)}\rangle \, . \tag{A.14}$$

Second order. We collect the terms of order ϵ^2 in (A.4):

$$H \, |\psi_i^{(2)}\rangle + \delta H \, |\psi_i^{(1)}\rangle = \lambda_i^{(0)} \, |\psi_i^{(2)}\rangle + \lambda_i^{(1)} \, |\psi_i^{(1)}\rangle + \lambda_i^{(2)} \, |\psi_i^{(0)}\rangle \, , \tag{A.15}$$

which we multiply by $\langle \psi_j^{(0)}|$ to obtain

$$\lambda_j^{(0)} \, \langle \psi_j^{(0)} | \psi_i^{(2)} \rangle + \langle \psi_j^{(0)}| \, \delta H \, |\psi_i^{(1)}\rangle = \lambda_i^{(0)} \, \langle \psi_j^{(0)} | \psi_i^{(2)} \rangle + \lambda_i^{(1)} \, \langle \psi_j^{(0)} | \psi_i^{(1)} \rangle + \lambda_i^{(2)} \delta_{ij}. \tag{A.16}$$

We calculate partial results by using first order equations for the corrections of the eigenvalue (A.8) and eigenvector (A.9):

$$\begin{aligned}
\langle \psi_j^{(0)}| \, \delta H \, |\psi_i^{(1)}\rangle &= \langle \psi_i^{(0)} | \psi_i^{(1)} \rangle \, \delta H_{ji} + \sum_{k(\neq i)} \frac{\delta H_{ki} \delta H_{jk}}{\lambda_i^{(0)} - \lambda_k^{(0)}}, \\
\lambda_i^{(1)} \, \langle \psi_j^{(0)} | \psi_i^{(1)} \rangle &= \frac{\delta H_{ii} \delta H_{ji}}{\lambda_i^{(0)} - \lambda_j^{(0)}}, \qquad i \neq j, \\
\lambda_i^{(1)} \, \langle \psi_i^{(0)} | \psi_i^{(1)} \rangle &= \delta H_{ii} \, \langle \psi_i^{(0)} | \psi_i^{(1)} \rangle \, ,
\end{aligned}$$

where we did not set the $\langle\psi_i^{(0)}|\psi_i^{(1)}\rangle = 0$ so far. We look instead at the case $i = j$ in (A.16) to obtain the second order correction $\lambda_i^{(2)}$ to the eigenvalue:

$$\begin{aligned}\lambda_i^{(2)} &= \langle\psi_i^{(0)}|\,\delta H\,|\psi_i^{(1)}\rangle - \lambda_i^{(1)}\,\langle\psi_i^{(0)}|\psi_i^{(1)}\rangle = \\ &= \sum_{k(\neq i)} \frac{\delta H_{ki}\delta H_{ik}}{\lambda_i^{(0)} - \lambda_k^{(0)}}.\end{aligned} \tag{A.17}$$

And with setting $i \neq j$ in (A.16) we find

$$\langle\psi_j^{(0)}|\psi_i^{(2)}\rangle = -\frac{\delta H_{ii}\delta H_{ji}}{\left(\lambda_i^{(0)} - \lambda_j^{(0)}\right)^2} + \langle\psi_i^{(0)}|\psi_i^{(1)}\rangle\,\frac{\delta H_{ji}}{\lambda_i^{(0)} - \lambda_j^{(0)}} + \sum_{k(\neq i)} \frac{\delta H_{ki}\delta H_{jk}}{\left(\lambda_i^{(0)} - \lambda_k^{(0)}\right)\left(\lambda_i^{(0)} - \lambda_j^{(0)}\right)},$$

so that the second order correction to the eigenvector reads

$$\begin{aligned}|\psi_i^{(2)}\rangle = \langle\psi_i^{(0)}|\psi_i^{(2)}\rangle\,|\psi_i^{(0)}\rangle + \sum_{l(\neq i)} \langle\psi_i^{(0)}|\psi_i^{(1)}\rangle\,\frac{\delta H_{ji}}{\lambda_i^{(0)} - \lambda_j^{(0)}}\,|\psi_l^{(0)}\rangle + \\ + \sum_{l(\neq i)} \left(-\frac{\delta H_{ii}\delta H_{ji}}{\left(\lambda_i^{(0)} - \lambda_j^{(0)}\right)^2} + \sum_{k(\neq i)} \frac{\delta H_{ki}\delta H_{jk}}{\left(\lambda_i^{(0)} - \lambda_k^{(0)}\right)\left(\lambda_i^{(0)} - \lambda_j^{(0)}\right)} \right) |\psi_l^{(0)}\rangle\,.\end{aligned}$$

We now specify the term $\langle\psi_i^{(0)}|\psi_i^{(2)}\rangle$ by inspecting the ϵ^2 correction to the orthogonality relation of (A.11):

$$\langle\psi_i^{(1)}|\psi_j^{(1)}\rangle + \langle\psi_i^{(2)}|\psi_j^{(0)}\rangle + \langle\psi_i^{(0)}|\psi_j^{(2)}\rangle = 0. \tag{A.18}$$

For $i \neq j$, after some algebra and by using $\langle\psi_i^{(1)}|\psi_i^{(0)}\rangle = 0$ we find it to be satisfied trivially. Consequently, by setting $i = j$ we have:

$$\mathrm{Re}\,\langle\psi_i^{(0)}|\psi_i^{(2)}\rangle = -\frac{1}{2}\sum_{k(\neq i)} \frac{\delta H_{ki}\delta H_{ik}}{\left(\lambda_i^{(0)} - \lambda_k^{(0)}\right)^2}. \tag{A.19}$$

By the same argument as before, imaginary part can be set to zero. The real part of (A.19) is dependent on the eigenvalues and δH_{ij} and thus cannot be set to zero. Finally, the second order correction to the eigenvector reads

$$\begin{aligned}|\psi_i^{(2)}\rangle = -\frac{1}{2}\sum_{k(\neq i)} \frac{\delta H_{ki}\delta H_{ik}}{\left(\lambda_i^{(0)} - \lambda_k^{(0)}\right)^2}\,|\psi_i^{(0)}\rangle + \\ + \sum_{l(\neq i)} \left(-\frac{\delta H_{ii}\delta H_{li}}{\left(\lambda_i^{(0)} - \lambda_l^{(0)}\right)^2} + \sum_{k(\neq i)} \frac{\delta H_{ki}\delta H_{lk}}{\left(\lambda_i^{(0)} - \lambda_k^{(0)}\right)\left(\lambda_i^{(0)} - \lambda_l^{(0)}\right)} \right) |\psi_l^{(0)}\rangle\,.\end{aligned} \tag{A.20}$$

We collect all the corrections (A.8), (A.14), (A.17) and (A.20) the eigenvalues and eigenvectors for up to the second order ϵ^2 read:

$$\delta\lambda_i = \epsilon\delta H_{ii} + \epsilon^2 \sum_{k(\neq i)} \frac{\delta H_{ki}\delta H_{ik}}{\lambda_i^{(0)} - \lambda_k^{(0)}} + O(\epsilon^3), \tag{A.21}$$

$$\begin{aligned}\delta\left|\psi_i\right\rangle =& \epsilon \sum_{k(\neq i)} \frac{\delta H_{ki}}{\lambda_i^{(0)} - \lambda_k^{(0)}} \left|\psi_k^{(0)}\right\rangle + \epsilon^2 \sum_{l,k(\neq i)} \frac{\delta H_{ki}\delta H_{lk}}{\left(\lambda_i^{(0)} - \lambda_k^{(0)}\right)\left(\lambda_i^{(0)} - \lambda_l^{(0)}\right)} \left|\psi_l^{(0)}\right\rangle + \\ &- \epsilon^2 \sum_{l(\neq i)} \frac{\delta H_{ii}\delta H_{li}}{\left(\lambda_i^{(0)} - \lambda_l^{(0)}\right)^2} \left|\psi_l^{(0)}\right\rangle - \frac{\epsilon^2}{2} \sum_{k(\neq i)} \frac{\delta H_{ki}\delta H_{ik}}{\left(\lambda_i^{(0)} - \lambda_k^{(0)}\right)^2} \left|\psi_i^{(0)}\right\rangle + O(\epsilon^3).\end{aligned} \tag{A.22}$$

The equations (2.19) and (2.23) are retrieved by setting $\epsilon = 1$.

Appendix B

Lamperti transformations

B.1 Derivation of the equations (2.51) and (2.52)

We derive a transformation $(z,\tau)\to(z',\tau')$ such that the ACP of (2.26) averaged over the dGUE in a harmonic potential and averaged over free dGUE coincide:

$$U_N(z;\tau)|_{\text{dGUE+OU}}=\alpha(z',\tau')U'_N(z';\tau')|_{\text{dGUE}}, \tag{B.1}$$

with a yet to be determined proportionality factor α. We express the jPDF (2.49) of dGUE+OU process with parameter functions

$$B(\tau)=\sqrt{\frac{1-e^{-2a\tau}}{2a}},\quad A(\tau)=e^{-a\tau} \tag{B.2}$$

as

$$P^a_\tau(H|H_0)=\frac{1}{c_a(\tau)}\exp\left(-\frac{N}{2B^2}\mathrm{Tr}(H-AH_0)^2\right). \tag{B.3}$$

with the normalization constant $c_a(\tau)=\left(B\sqrt{\frac{2\pi}{N}}\right)^{N^2}$. The ACP on the l.h.s. of (B.1) reads

$$U_N(z;\tau)|_{\text{dGUE+OU}}=\int d[H]P^a_\tau(H|H_0)\det(z-H). \tag{B.4}$$

We change the variables $H=H_0A+B\mathcal{H}$, so that the matrix measure is transformed $d[H]=B^{N^2}d[\mathcal{H}]$ and $B^{-2}\mathrm{Tr}(H_\tau-H_0A)^2=\mathrm{Tr}\mathcal{H}^2$:

$$\begin{aligned}U_N(z;\tau)|_{\text{dGUE+OU}}&=\frac{B^{N^2}}{c_a(\tau)}\int d[\mathcal{H}]\exp\left(-\frac{N}{2}\mathrm{Tr}\mathcal{H}^2\right)\det(z-H_0A-B\mathcal{H})=\\&=\frac{B^{N^2}A^N}{c_a(\tau)}\int d[\mathcal{H}]\exp\left(-\frac{N}{2}\mathrm{Tr}\mathcal{H}^2\right)\det(A^{-1}z-H_0-A^{-1}B\mathcal{H}).\end{aligned}$$

Now we set new variables (z',τ') in the following way:

$$\tau'=A^{-2}B^2,\quad z'=A^{-1}z, \tag{B.5}$$

so that

$$U_N(z;\tau)|_{\text{dGUE+OU}}=\frac{B^{N^2}A^N}{c_a(\tau)}\int d[\mathcal{H}]\exp\left(-\frac{N}{2}\mathrm{Tr}\mathcal{H}^2\right)\det\left(z'-H_0-\sqrt{\tau'}\mathcal{H}\right), \tag{B.6}$$

and finally, by rescaling again $H_0 + \sqrt{\tau'}\mathcal{H} = H'$, we calculate $(\sqrt{\tau'})^{N^2} d[\mathcal{H}] = d[H']$ and we arrive at

$$U_N(z;\tau)|_{\text{dGUE+OU}} = \frac{A^{N(N+1)}}{c_a(\tau)} \int d[H'] \exp\left(-\frac{N}{2\tau'}\mathrm{Tr}(H'-H_0)^2\right) \det(z'-H'). \quad \text{(B.7)}$$

Lastly, as this transformed jPDF should be normalized:

$$\int d[H'] \exp\left(-\frac{N}{2\tau'}\mathrm{Tr}(H'-H_0)^2\right) = c'(\tau') = c_a(\tau)A^{-N^2}, \quad \text{(B.8)}$$

we obtain $c'(\tau') = \left(\sqrt{\frac{2\pi\tau'}{N}}\right)^{N^2}$. Finally, the l.h.s. of (B.1) reads

$$\begin{aligned} U_N(z;\tau)|_{\text{dGUE+OU}} &= A^N \frac{1}{c'(\tau')} \int d[H'] \exp\left(-\frac{N}{2\tau'}\mathrm{Tr}(H'-H_0)^2\right) \det(z'-H') \equiv \\ &\equiv A^N U'_N(z';\tau')|_{\text{dGUE}}, \end{aligned} \quad \text{(B.9)}$$

where $U'_N(z';\tau')|_{\text{dGUE}}$ is a normalized average over the jPDF of a free dGUE of (2.17) and the proportionality factor of (B.1) reads $\alpha(z',\tau') = A^N$. Out of (B.5), we calculate the factor $A(\tau') = (1+2a\tau')^{-1/2}$.

To conclude, the Lamperti transformation reads:

$$U_N(z;\tau) = (1+2a\tau')^{-N/2} U'_N(z';\tau'), \quad \text{(B.10)}$$

$$z' = e^{a\tau} z, \quad \tau' = \frac{1}{2a}\left(e^{2a\tau} - 1\right). \quad \text{(B.11)}$$

To make sure that U'_N fulfils a free diffusion equation, we define $f = (1+2a\tau')^{-N/2}$, and calculate partial results

$$\partial_\tau U_N = A^{-2}\left(-\frac{Na}{1+2a\tau'} f U'_N + f\partial_{\tau'} U'_N\right) + az' f \partial_{z'} U'_N, \quad \text{(B.12)}$$

$$\partial_z U_N = f A^{-1} \partial_{z'} U'_N, \quad \text{(B.13)}$$

$$\partial_{zz} U_N = f A^{-2} \partial_{z'z'} U'_N, \quad \text{(B.14)}$$

by which we transform the formula (2.50)

$$\partial_\tau U_N(z;\tau) = -\frac{1}{2N}\partial_{zz} U_N(z;\tau) + az\partial_z U_N(z;\tau) - aN U_N(z;\tau) \quad \text{(B.15)}$$

to a free diffusion of (2.28):

$$\partial_{\tau'} U'_N = -\frac{1}{2N}\partial_{z'z'} U'_N. \quad \text{(B.16)}$$

B.2 Derivation of the equations (3.59) and (3.60)

We consider Lamperti transformation $(Q,\tau) \to (Q',\tau')$ such that the AECP transforms in the following way:

$$D_N(Q;\tau)|_{\text{dGinUE+OU}} = \beta(Q',\tau') D'_N(Q';\tau')|_{\text{dGinUE}}, \quad \text{(B.17)}$$

where D_N is defined in (3.18). The jPDF for the dGinUE+OU of (3.57) is formulated as:

$$P_\tau^a(X|X_0) = \frac{1}{D'(\tau)} \exp\left(-NB^{-2}\mathrm{Tr}|X - X_0A|^2\right), \tag{B.18}$$

with the A, B parameter functions defined in (B.2) and normalization constant $D'(\tau) = \left(B\sqrt{\frac{\pi}{N}}\right)^{2N^2}$. The l.h.s. of (B.17) reads

$$\text{l.h.s.} = \int d[X] P_\tau^a(X|X_0) \det(Q - \mathcal{X}), \tag{B.19}$$

where $Q, \mathcal{X}$ were defined in (3.6). Let $\mathcal{X} = \mathcal{X}_0 A + B\mathcal{Y}$, the measure is then transformed $d[X] = B^{2N^2} d[Y]$ and we find $B^{-2}\mathrm{Tr}|X - X_0A|^2 = \mathrm{Tr}Y^\dagger Y$. We continue with the l.h.s. of (B.17):

$$\begin{aligned}\text{l.h.s.} &= \frac{B^{2N^2}}{D'(\tau)} \int d[Y] \exp\left(-N\mathrm{Tr}Y^\dagger Y\right) \det(Q - A\mathcal{X}_0 - B\mathcal{Y}) = \\ &= \frac{B^{2N^2}A^{2N}}{D'(\tau)} \int d[Y] \exp\left(-N\mathrm{Tr}Y^\dagger Y\right) \det(A^{-1}Q - \mathcal{X}_0 - A^{-1}B\mathcal{Y}).\end{aligned}$$

Now we set new variables (τ', Q') in the following way:

$$\tau' = A^{-2}B^2, \quad Q' = A^{-1}Q, \tag{B.20}$$

so that

$$\text{l.h.s.} = \frac{B^{2N^2}A^{2N}}{D'(\tau)} \int d[Y] \exp\left(-N\mathrm{Tr}^\dagger Y\right) \det\left(Q' - \mathcal{X}_0 - \sqrt{\tau'}\mathcal{Y}\right), \tag{B.21}$$

and finally, by rescaling again $\mathcal{X}_0 + \sqrt{\tau'}\mathcal{Y} = \mathcal{X}'$ and computing $(\sqrt{\tau'})^{2N^2} d[X] = d[X']$, we arrive at

$$\text{l.h.s.} = \frac{A^{2N(N+1)}}{D'(\tau)} \int d[X'] \exp\left(-\frac{N}{\tau'}\mathrm{Tr}|X' - X_0|^2\right) \det(Q' - \mathcal{X}'). \tag{B.22}$$

Lastly, this transformed jPDF should be normalized

$$\int d[X'] \exp\left(-\frac{N}{\tau'}\mathrm{Tr}|X' - X_0|^2\right) = D'' = D'A^{-2N^2} \tag{B.23}$$

which gives $D'' = \left(\sqrt{\frac{\pi\tau'}{N}}\right)^{2N^2}$. Finally, the l.h.s. reads

$$\begin{aligned}\text{l.h.s.} &= A^{2N}\frac{1}{D''} \int d[X'] \exp\left(-\frac{N}{\tau'}\mathrm{Tr}|X' - X_0|^2\right) \det(Q' - \mathcal{X}') \equiv \\ &\equiv A^{2N} D'_N(Q'; \tau')|_{\text{dGinUE}},\end{aligned} \tag{B.24}$$

where $D'_N(Q', \tau')|_{\text{dGinUE}}$ is averaged over the jPDF of free dGinUE (3.2) and the proportionality factor of (B.17) reads $\beta = A^{2N}$. To conclude, we write down the Lamperti

transformation:

$$D_N(Q;\tau) = (1+2a\tau')^{-N} D'_N(Q';\tau'), \tag{B.25}$$

$$Q' = e^{a\tau}Q, \quad \tau' = \frac{1}{2a}\left(e^{2a\tau}-1\right). \tag{B.26}$$

where we calculate $A(\tau') = (1+2a\tau')^{-1/2}$.

To check that indeed D'_N fulfils a free diffusion equation, we define $f = (1+2a\tau')^{-N}$, and calculate partial results

$$\partial_\tau D_N = -2Nfa D'_N + fA^{-2}\partial_{\tau'} D'_N + faA^{-1}\mathrm{d}D'_N, \tag{B.27}$$

$$\partial_{Q_{\alpha\beta}} D_N = fA^{-1}\partial_{Q'_{\alpha\beta}} D'_N, \quad \{\alpha,\beta\} = 1,2, \tag{B.28}$$

$$\partial_{w\bar{w}} D_N = fA^{-2}\partial_{w'\bar{w}'} D'_N, \tag{B.29}$$

by which, after plugging into (3.58), we recover the free diffusion

$$\partial_{\tau'} D'_N = \frac{1}{N}\partial_{w'\bar{w}'} D'_N. \tag{B.30}$$

Appendix C

Various calculations

C.1 Proof of the formula (2.66)

We present a simple proof of the relation:

$$\partial_x \int_I f(x') \log|x - x'| dx' = \mathrm{PV} \int_I \frac{f(x')}{x - x'} dx', \qquad x \in I. \tag{C.1}$$

Let $I = (a, b)$ and consider l.h.s. of (C.1):

$$\text{l.h.s.} = \lim_{\epsilon \to 0_+} \partial_x \left(\int_a^{x-\epsilon} dx' f(x') \log|x - x'| + \int_{x+\epsilon}^{b} dx' f(x') \log|x - x'| \right). \tag{C.2}$$

By splitting the integral we get rid of absolute value and so

$$\text{l.h.s.} = \lim_{\epsilon \to 0_+} \left(\int_a^{x-\epsilon} dx' f(x') \frac{1}{x - x'} + \int_{x+\epsilon}^{b} dx' f(x') \frac{1}{x - x'} + (f(x - \epsilon) - f(x + \epsilon)) \log \epsilon \right).$$

We notice how first two terms comprise the principal value:

$$\text{l.h.s.} = \mathrm{PV} \int_a^b \frac{f(x')}{x - x'} dx' + C, \tag{C.3}$$

whereas the second term C:

$$C = \lim_{\epsilon \to 0_+} (f(x - \epsilon) - f(x + \epsilon)) \log \epsilon = 0 \tag{C.4}$$

vanishes for sufficiently well behaved f.

C.2 Derivations of the equation (2.123)

The starting point is the functional SFP equation (2.120):

$$\partial_t \hat{\Pi}_t = (\hat{K} + \hat{V}) \hat{\Pi}_t. \tag{C.5}$$

As a first step, we calculate the r.h.s. of (C.5) — operators $\hat{K}$ and $\hat{V}$ of (2.119) acting on the ansatz (2.122):

$$\hat{\Pi}_t \left[\hat{\lambda} | \hat{\lambda}^0 \right] = \exp\left(-\frac{\beta}{2} N^2 S_t \left[\hat{\lambda} | \hat{\lambda}^0 \right] \right). \tag{C.6}$$

The $\hat{K}$ acts on $\hat{\Pi}_t$ to produce:

$$\mathrm{K} \equiv \left(-\frac{\beta N^2}{2}\right)^{-1} (\hat{\Pi}_t)^{-1} \hat{K} \hat{\Pi}_t = \frac{1}{\beta} \int dq \sum_{i=1}^{N} \frac{\partial^2 \hat{\lambda}(q)}{\partial \lambda_i^2} \frac{\delta S_t}{\delta \hat{\lambda}(q)} + \\ + \frac{1}{\beta} \int dpdq \sum_{i=1}^{N} \frac{\partial \hat{\lambda}(q)}{\partial \lambda_i} \frac{\partial \hat{\lambda}(p)}{\partial \lambda_i} \left(\frac{\delta^2 S_t}{\delta \hat{\lambda}(p) \delta \hat{\lambda}(q)} - \frac{\beta N^2}{2} \frac{\delta S_t}{\delta \hat{\lambda}(p)} \frac{\delta S_t}{\delta \hat{\lambda}(q)} \right), \tag{C.7}$$

and likewise the operator $\hat{V}$ acts on $\hat{\Pi}_t$ to produce:

$$\mathrm{V} \equiv \left(-\frac{\beta N^2}{2}\right)^{-1} (\hat{\Pi}_t)^{-1} \hat{V} \hat{\Pi}_t = - \int dq \left(\sum_{i=1}^{N} \frac{\partial W}{\partial \lambda_i} \frac{\partial \hat{\lambda}(q)}{\partial \lambda_i} \right) \frac{\delta S_t}{\delta \hat{\lambda}(q)}. \tag{C.8}$$

On the other hand, acting with ∂_t on $\hat{\Pi}_t$ produces

$$\left(-\frac{\beta N^2}{2}\right)^{-1} (\hat{\Pi}_t)^{-1} \partial_t \hat{\Pi}_t = \partial_t S_t \tag{C.9}$$

The first part of (C.7), with the help of the formula $\partial_{\lambda_i} \hat{\lambda} = -\partial_q \delta(q - \lambda_i)$, reads

$$\frac{1}{\beta} \int dq \left(\sum_i \frac{\partial^2 \hat{\lambda}(q)}{\partial \lambda_i^2} \right) \frac{\delta S_t}{\delta \hat{\lambda}(q)} = \frac{1}{\beta} \int dp \hat{\lambda}(p) \partial_{pp} \frac{\delta S_t}{\delta \hat{\lambda}(p)}. \tag{C.10}$$

The second and third term of K in turn is equal to

$$\frac{1}{\beta} \int dpdq \left(\sum_i \frac{\partial \hat{\lambda}(q)}{\partial \lambda_i} \frac{\partial \hat{\lambda}(p)}{\partial \lambda_i} \right) \frac{\delta^2 S_t}{\delta \hat{\lambda}(p) \delta \hat{\lambda}(q)} = \frac{1}{\beta} \int dp \hat{\lambda}(p) \partial_{pp} \frac{\delta^2 S_t}{\delta \hat{\lambda}(p)^2}, \\ -\frac{N^2}{2} \int dpdq \left(\sum_i \frac{\partial \hat{\lambda}(q)}{\partial \lambda_i} \frac{\partial \hat{\lambda}(p)}{\partial \lambda_i} \right) \frac{\delta S_t}{\delta \hat{\lambda}(p)} \frac{\delta S_t}{\delta \hat{\lambda}(q)} = -\frac{N^2}{2} \int dp \hat{\lambda}(p) \left(\partial_p \frac{\delta S_t}{\delta \hat{\lambda}(p)} \right)^2,$$

and so K reads

$$\mathrm{K} = \frac{1}{\beta} \int dp \hat{\lambda}(p) \partial_{pp} \frac{\delta S_t}{\delta \hat{\lambda}(p)} - \frac{N^2}{2} \int dp \hat{\lambda}(p) \left(\partial_p \frac{\delta S_t}{\delta \hat{\lambda}(p)} \right)^2 + \frac{1}{\beta} \int dp \hat{\lambda}(p) \partial_{pp} \frac{\delta^2 S_t}{\delta \hat{\lambda}(p)^2}. \tag{C.11}$$

The potential part V is calculated with the help of

$$\frac{1}{\lambda_i - \lambda_j} = \mathrm{PV} \int d\mu \frac{1}{\lambda_i - \mu} \delta(\lambda_j - \mu), \\ \sum_{i \neq j} \delta(p - \lambda_i) \delta(q - \lambda_j) = \hat{\lambda}(p) \hat{\lambda}(q) - \delta(p - q) \hat{\lambda}(p),$$

where the first is a special case of (2.66). The potential part V reads

$$\begin{aligned}
\mathrm{V} &= -\int dq \left(\sum_i \frac{\partial W}{\partial \lambda_i} \frac{\partial \hat{\lambda}(q)}{\partial \lambda_i} \right) \frac{\delta S_t}{\delta \hat{\lambda}(q)} = -\int dq \left(\sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \partial_q \delta(q - \lambda_i) \right) \frac{\delta S_t}{\delta \hat{\lambda}(q)} = \\
&= \int dq \mathrm{PV} \int d\mu \sum_{i \neq j} \frac{1}{\lambda_i - \mu} \delta(\lambda_j - \mu) \delta(\lambda_i - q) \partial_q \frac{\delta S_t}{\delta \hat{\lambda}(q)} = \\
&= \int dq \hat{\lambda}(q) \left(\mathcal{H}[\hat{\lambda}](q) - PV \int d\mu \frac{\delta(q-\mu)}{q-\mu} \right) \partial_q \frac{\delta S_t}{\delta \hat{\lambda}(q)} = \\
&= \int dq \hat{\lambda}(q) \mathcal{H}[\hat{\lambda}](q) \partial_q \frac{\delta S_t}{\delta \hat{\lambda}(q)} - \mathcal{H}[\delta](0) \int dq \hat{\lambda}(q) \partial_q \frac{\delta S_t}{\delta \hat{\lambda}(q)},
\end{aligned} \tag{C.12}$$

where we used the Hilbert transform $\mathcal{H}$ of (2.65).

By collecting (C.9), (C.11) and (C.12), the functional SFP equation of (2.120) reads

$$\begin{aligned}
\partial_t S_t = &\int dp\, \hat{\lambda}(p) \left(\frac{1}{\beta} \frac{\partial^2}{\partial p^2} \left(\frac{\delta S_t}{\delta \hat{\lambda}(p)} + \frac{\delta^2 S_t}{\delta \hat{\lambda}(p)^2} \right) - \mathcal{H}[\delta](0) \frac{\partial}{\partial p} \frac{\delta S_t}{\delta \hat{\lambda}(p)} \right) + \\
&- \int dp\, \hat{\lambda}(p) \left(\frac{N^2}{2} \left(\frac{\partial}{\partial p} \frac{\delta S_t}{\delta \hat{\lambda}(p)} \right)^2 - \mathcal{H}[\hat{\lambda}](p) \frac{\partial}{\partial p} \frac{\delta S_t}{\delta \hat{\lambda}(p)} \right),
\end{aligned} \tag{C.13}$$

which is exactly the equation (2.123).

C.3 Derivation of the equation (2.131)

We start from the functional SFP equation of (2.120) with a replacement $\hat{\Pi}_t \to \hat{\Pi}^{\mathrm{c}}_t$:

$$\partial_t \hat{\Pi}^{\mathrm{c}}_t = (\hat{K} + \hat{V}) \hat{\Pi}^{\mathrm{c}}_t, \tag{C.14}$$

In the definitions (2.119) of the operators $\hat{K}$ and $\hat{V}$ we replace $\lambda \to x$ and read off the W potential from the equations (2.60) and (2.75). The collective variable reads

$$\hat{x}(q) = \sum_{i=1}^{m} \Big(\delta(q - x_i) + \delta(q + x_i) \Big) = \sum_{i=1}^{m} 2|q| \delta(q^2 - x_i^2), \tag{C.15}$$

and we consider the ansatz

$$\hat{\Pi}^{\mathrm{c}}_t = \exp\left(-\frac{\beta}{4} m^2 S^{\mathrm{c}}_t \right), \tag{C.16}$$

Action of $\hat{K}$ and $\hat{V}$ on $\hat{\Pi}_t^{\mathrm{c}}$ produces the formulas (C.7) and (C.8) with the replacements $\beta \to \frac{\beta}{2}$, $N \to m$, $S_t \to S_t^{\mathrm{c}}$ and $\hat{\lambda} \to \hat{x}$. Firstly, we compute partial results

$$\begin{aligned}
&\frac{\partial}{\partial x_j}\left(\sum_{i=1}^{m} 2|q|\delta(q^2-x_i^2)\right) = -\frac{|q|}{q}\frac{\partial}{\partial q}\Big(2|q|\delta(q^2-x_j^2)\Big),\\
&\frac{1}{x_j}\delta(q^2-x_j^2) = \mathrm{PV}\int_{-\infty}^{\infty}\frac{d\mu\,\delta(\mu)}{|q|-\mu}\delta(q^2-x_j^2),\\
&\frac{2x_j}{x_j^2-x_k^2}\delta(q^2-x_j^2) = \mathrm{PV}\int_{-\infty}^{\infty}\frac{d\mu 2|\mu|\,\delta(\mu^2-x_k^2)}{|q|-\mu}\delta(q^2-x_j^2),\\
&\sum_{j\neq i} 2|p|\delta(p^2-x_i^2)2|q|\delta(q^2-x_j^2) = \hat{x}(p)\hat{x}(q) - \delta(p-q)\hat{x}(q).
\end{aligned}$$

So that the parts of $\mathrm{K} \equiv \left(-\frac{\beta m^2}{4}\right)^{-1}(\hat{\Pi}_t^{\mathrm{c}})^{-1}\hat{K}\hat{\Pi}_t^{\mathrm{c}}$ are equal to

$$\begin{aligned}
&\frac{1}{\beta}\int dq\left(\sum_i \frac{\partial^2\hat{x}(q)}{\partial x_i^2}\right)\frac{\delta S_t^{\mathrm{c}}}{\delta\hat{x}(q)} = \frac{1}{\beta}\int dq\hat{x}(q)\partial_{qq}\frac{\delta S_t^{\mathrm{c}}}{\delta\hat{x}(q)},\\
&\frac{1}{\beta}\int dpdq\left(\sum_i \frac{\partial\hat{x}(q)}{\partial x_i}\frac{\partial\hat{x}(p)}{\partial x_i}\right)\frac{\delta^2 S_t^{\mathrm{c}}}{\delta\hat{x}(p)\delta\hat{x}(q)} = \frac{2}{\beta}\int dp\hat{x}(p)\partial_{pp}^2\frac{\delta^2 S_t^{\mathrm{c}}}{\delta\hat{x}(p)^2},\\
&-\frac{m^2}{4}\int dpdq\left(\sum_i \frac{\partial\hat{x}(q)}{\partial x_i}\frac{\partial\hat{x}(p)}{\partial x_i}\right)\frac{\delta S_t^{\mathrm{c}}}{\delta\hat{x}(p)}\frac{\delta S_t^{\mathrm{c}}}{\delta\hat{x}(q)} = -\frac{m^2}{2}\int dp\hat{x}(p)\left(\partial_p\frac{\delta S_t^{\mathrm{c}}}{\delta\hat{x}(p)}\right)^2
\end{aligned}$$

and together form

$$\mathrm{K} = \frac{1}{\beta}\int dq\hat{x}(q)\partial_{qq}\frac{\delta S_t^{\mathrm{c}}}{\delta\hat{x}(q)} + \frac{2}{\beta}\int dp\hat{x}(p)\partial_{pp}^2\frac{\delta^2 S_t^{\mathrm{c}}}{\delta\hat{x}(p)^2} - \frac{m^2}{2}\int dp\hat{x}(p)\left(\partial_p\frac{\delta S_t^{\mathrm{c}}}{\delta\hat{x}(p)}\right)^2. \tag{C.17}$$

In a similar manner, the $\mathrm{V} \equiv \left(-\frac{\beta m^2}{4}\right)^{-1}(\hat{\Pi}_t^{\mathrm{c}})^{-1}\hat{V}\hat{\Pi}_t^{\mathrm{c}}$ reads

$$\begin{aligned}
\mathrm{V} = -\int dq\left(\sum_i \frac{\partial W}{\partial x_i}\frac{\partial\hat{x}(q)}{\partial x_i}\right)\frac{\delta S_t^{\mathrm{c}}}{\delta\hat{x}(q)} = \int dp\hat{x}(p)\mathcal{H}[a'\delta+\hat{x}](p)\partial_p\left(\frac{\delta S_t^{\mathrm{c}}}{\delta\hat{x}(p)}\right) +\\
+\int dp\,\hat{x}(p)\mathcal{H}[\delta](0)\frac{\partial}{\partial p}\frac{\delta S_t^{\mathrm{c}}}{\delta\hat{x}(p)},
\end{aligned} \tag{C.18}$$

and the SFP equation is finally given as

$$\begin{aligned}
\partial_t S_t^{\mathrm{c}} = \int dp\,\hat{x}(p)\left(\frac{1}{\beta}\frac{\partial^2}{\partial p^2}\left(\frac{\delta S_t^{\mathrm{c}}}{\delta\hat{x}(p)} + 2\frac{\delta^2 S_t^{\mathrm{c}}}{\delta\hat{x}(q)^2}\right) + \mathcal{H}[\delta](0)\frac{\partial}{\partial p}\frac{\delta S_t^{\mathrm{c}}}{\delta\hat{x}(p)}\right) +\\
-\int dp\,\hat{x}(p)\left(\frac{m^2}{2}\left(\frac{\partial}{\partial p}\frac{\delta S_t^{\mathrm{c}}}{\delta\hat{x}(p)}\right)^2 - \mathcal{H}[a'\delta+\hat{x}](p)\frac{\partial}{\partial p}\frac{\delta S_t^{\mathrm{c}}}{\delta\hat{x}(p)}\right),
\end{aligned} \tag{C.19}$$

which is exactly the formula of (2.131).

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Dysonian Dynamics of the Ginibre Ensemble

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(Received 11 April 2014; published 4 September 2014)

We study the time evolution of Ginibre matrices whose elements undergo Brownian motion. The non-Hermitian character of the Ginibre ensemble binds the dynamics of eigenvalues to the evolution of eigenvectors in a nontrivial way, leading to a system of coupled nonlinear equations resembling those for turbulent systems. We formulate a mathematical framework allowing simultaneous description of the flow of eigenvalues and eigenvectors, and we unravel a hidden dynamics as a function of a new complex variable, which in the standard description is treated as a regulator only. We solve the evolution equations for large matrices and demonstrate that the nonanalytic behavior of the Green's functions is associated with a shock wave stemming from a Burgers-like equation describing correlations of eigenvectors. We conjecture that the hidden dynamics that we observe for the Ginibre ensemble is a general feature of non-Hermitian random matrix models and is relevant to related physical applications.

DOI: 10.1103/PhysRevLett.113.104102 PACS numbers: 05.45.Mt, 02.10.Yn, 02.50.Ey, 05.10.-a

Today, half a century after the pioneering work of Ginibre [1], random matrices with complex spectra are no longer only of academic interest. They play a role in quantum chaotic scattering [2,3], quantum information processing [4], QCD with finite chemical potential [5], in financial engineering with lagged correlations [6], and in the research on neural networks [7], to name just a few applications. Eigenvalues themselves, however, are not of sole interest in the case of non-Hermitian random matrix ensembles. The statistical properties of eigenvectors are equally significant [8], in particular, in problems concerning scattering in open chaotic cavities or random lasing [9–12]. There, the so called Petermann factor [13], a quantity describing correlations between right and left eigenvectors, modifies the quantum-limited linewidth of a laser.

On the other hand, the original Dyson's idea of a Brownian walk of real eigenvalues [14] interacting with a two-dimensional Coulombic force still leads to novel insights. Examples include the study of determinantal processes [15–17], Loewner diffusion [18], non-Hermitian deformations [19], or the fluctuations of nonintersecting interfaces in thermal equilibrium [20]. The concept of matricial stochastic evolution has been recently exploited by several authors [21–24]. In particular, it was shown that the derivatives of the logarithms of characteristic determinants of diffusing GUE (Gaussian unitary ensemble), LUE (Laguerre unitary ensemble) and CUE (Circular unitary ensemble) obey Burgers-like nonlinear equations, where the role of viscosity is played by the inverse of the matrix size. For infinite dimensions of the matrix, these equations correspond to the inviscid regime and describe evolution of the associated resolvents. Because of nonlinearity, they develop singularities (shock waves), whose positions correspond to the endpoints of the spectra. For matrices of finite size, the expansion around the shock wave solution of the initial viscid Burgers equation leads to a universal scaling of characteristic polynomials (and of the inverse characteristic polynomials as well), resulting in well-known universal oscillatory behavior of the Airy, Bessel, or Pearcey type. This approach has prompted, in particular, new perception of weak or strong coupling transition in multicolor Yang-Mills theory [25,26] and of the spontaneous breakdown of chiral symmetry in Euclidean QCD [27].

In this Letter, we unveil the intertwined evolution of eigenvalues and eigenvectors of stochastically evolving non-Hermitian matrices. To this end, we apply Dyson's idea to study diffusing matrices for the case of the Ginibre ensemble (GE). The central object of the Letter is a generalized averaged characteristic polynomial. Its logarithmic derivatives, which contain the information about both the eigenvalues and eigenvectors of the evolving matrix, fulfill a system of Burgers-like partial differential equations. We solve them to recover the spectral density, the Petermann factor encoding the correlations of eigenvectors and universal microscopic scaling at the edge of the support of the eigenvalues.

At first glance one would not expect any similarities between the GUE and the GE, even in the large N (matrix size) limit. In the case of GUE, spectra are real; end points of the spectra exhibit square root behavior and the eigenvectors decouple from the eigenvalues. In the case of GE, spectra are complex, eigenvalues form a uniform disc with a vertical cliff at the boundary and the eigenvectors are correlated [8] on the support of eigenvalues. Nonetheless, the Vandermonde determinant is present in the joint probability distribution of eigenvalues for both ensembles and this leads to a two-dimensional electrostatic Dyson's picture that underlies calculations of the spectral

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distribution in the large N limit. Consequently, the standard procedure for non-Hermitian ensembles relies on defining the electrostatic potential

$$V(z) = \lim_{\epsilon\to 0}\lim_{N\to\infty}\frac{1}{N}\langle \mathrm{Tr}\,\ln[|z-X|^2+\epsilon^2]\rangle, \tag{1}$$

calculating the "electric field" as its gradient $G=\partial_z V$, and recovering the spectral function from the Gauss law $\rho=(1/\pi)\partial_{\bar z}G=(1/\pi)\partial_{z\bar z}V$. We use a shorthand notation defined by $|z-X|^2+\epsilon^2=(z\mathbb{1}_N-X)(\bar z\mathbb{1}_N-X^\dagger)+\epsilon^2\mathbb{1}_N$, where $\mathbb{1}_N$ is the N-dimensional identity matrix. ϵ is an infinitesimal regulator and it is crucial that the limit $N\to\infty$ is taken first. If one took the limits in an opposite order, one would obtain a trivial result. Moreover, in the case of the Ginibre ensemble, $\langle\det(z-X)\rangle = z^N$. The standard relation between zeros of the characteristic polynomials and poles of the Green's function, known from considerations of Hermitian ensembles, would therefore be lost.

The idea is to define the following object

$$\begin{aligned} D(z,w,\tau) &= \langle\det(Q-H)\rangle_\tau \\ &= \langle\det\left(|z-X|^2+|w|^2\right)\rangle_\tau, \end{aligned} \tag{2}$$

where

$$Q=\begin{pmatrix} z & -\bar w\\ w & \bar z\end{pmatrix},\qquad H=\begin{pmatrix} X & 0\\ 0 & X^\dagger\end{pmatrix} \tag{3}$$

and to study its evolution in the space of Q, or more precisely in the complex plane w "perpendicular" to the basic complex plane z. In other words, the regulator $i\epsilon$, which is usually treated as an infinitesimally small real variable, is promoted to a genuine complex variable w. Note that D is effectively a characteristic determinant expressed in terms of the quaternion variable Q, since $Q=q_0+i\sigma_j q_j$, where σ_j are Pauli matrices, so $z=q_0+iq_3$ and $w=q_1+iq_2$. As we shall see, the dynamics of $D(z,w,\tau)$ hidden in w captures the evolution of eigenvectors and eigenvalues of the Ginibre matrix whose elements undergo Brownian motion. It is worth mentioning that block matrices such as H and arguments Q naturally appear in non-Hermitian random matrix models, e.g., in the generalized Green's function technique [28,29], in Hermitization methods [30–32], in the derivation of the multiplication law for non-Hermitian random matrices [33], and in the weak non-Hermitian random ensembles [34].

In our notation, the meaning of the averages $\langle\ldots\rangle_\tau$ like this in (2) is $\langle F(X)\rangle_\tau=\int DX P(X,\tau|X_0,0)F(X)$, where $DX=\sum_{ab}dx_{ab}dy_{ab}$ is a flat measure over the real and imaginary parts of matrix elements, $X_{ab}=x_{ab}+iy_{ab}$, and $P(X,\tau|X_0,0)$ is the probability that the matrix will change from its initial state X_0 at $\tau=0$ to X at time τ. For a free random walk with independent increments $\langle\delta X_{ab}\rangle_\tau=0$ and $\langle\delta X_{ab}\delta\bar X_{cd}\rangle_\tau=(\delta\tau/N)\delta_{ac}\delta_{bd}$, the evolution of $P(X,\tau|X_0,0)$ is governed by the diffusion equation

$$\partial_\tau P(X,\tau|X_0,0)=\frac{1}{N}\partial_{XX^\dagger}P(X,\tau|X_0,0), \tag{4}$$

where $\partial_{XX^\dagger}$ is the standard $2N^2$-dimensional Laplacian $\partial_{XX^\dagger}=\sum_{ab}(\partial^2_{x_{ab}}+\partial^2_{y_{ab}})$. The announced dynamics of the Ginibre ensemble is hidden in equation

$$\partial_\tau D(z,w,\tau)=\frac{1}{N}\partial_{w\bar w}D(z,w,\tau), \tag{5}$$

which is central to this Letter. The derivation will be presented elsewhere, but we shortly sketch below the main steps. The determinant in (2) can be represented as a Berezin integral $\int\exp[\theta^T(Q-H)\eta]d\theta d\eta=\det(Q-H)$, where θ and η are independent vectors of Grassmann variables. Both sides of Eq. (4) can be then multiplied by this integral and integrated over DX. After some manipulations, like changing the order of integration and integrating by parts, one arrives at (5).

It is easy to see that $D(z,w,\tau)$ depends on w only through its modulus $r=|w|$. Moreover, the simplest initial condition corresponds to $X_0=0$ with $D_0(z,w)=D(z,w,0)=(|z|^2+|w|^2)^N$. The general matrix X_0 is determined by the eigenvalues Λ and a set of left-(L) and right-(R) eigenvectors $X_0R=R\Lambda$ ($L^\dagger X_0=\Lambda L^\dagger$). By applying a transformation $S=\mathrm{diag}(R,L)$, $S^{-1}=\mathrm{diag}(L^\dagger,R^\dagger)$, the off-diagonal blocks depend explicitly on the eigenvectors

$$\det\left(S^{-1}(Q-H)S\right)=\det\begin{pmatrix} z-\Lambda & -\bar wL^\dagger L\\ wR^\dagger R & \bar z-\Lambda^\dagger\end{pmatrix}. \tag{6}$$

This calculation shows that nonzero w indeed encodes full information of the underlying matrix which turns out to be valuable in what follows.

We define two convenient functions $v=v(z,r,\tau)$ and $g=g(z,r,\tau)$:

$$v\equiv\frac{1}{2N}\partial_r\ln D, \tag{7}$$

$$g\equiv\frac{1}{N}\partial_z\ln D, \tag{8}$$

which will turn out to be closely related to the eigenvector correlator and the Green's function known from the standard treatment of the Ginibre ensemble. These functions are not independent, since by construction $\partial_z v=\frac{1}{2}\partial_r g$; in particular, $g=2\int dr\partial_z v$. The diffusion equation (5) is mapped via (7), which basically is the inverse Cole-Hopf transformation [35], onto a Burgers-like equation

$$\partial_\tau v=v\partial_r v+\frac{1}{N}\left(\Delta_r-\frac{1}{4r^2}\right)v, \tag{9}$$

where $\Delta_r=\frac{1}{4}(\partial_{rr}+(1/r)\partial_r)$ is the radial part of the two-dimensional Laplacian. This equation is exact for any N.

The $1/N$ factor is a viscosity-like parameter. In the inviscid limit ($N \to \infty$), (9) reduces to

$$\partial_\tau v = v\partial_r v, \tag{10}$$

known as the Euler equation and solved by the method of characteristics. The curves along which the solution is constant are given by

$$r = \xi - v_0(\xi)\tau, \tag{11}$$

and labeled with ξ. v_0 plays the role of velocity of the front wave. We therefore have

$$v = v_0(r + \tau v). \tag{12}$$

For the initial condition $X_0 = 0$, corresponding to $v_0(r) = r/(z\bar{z} + r^2)$, we obtain a cubic algebraic equation for v. Its solution gives the (radial) dependence of v on $r = |w| \geq 0$. If one takes a cross section of the whole solution along the real axis, $\mathrm{Im}w = 0$ and $\mathrm{Re}w = \mu$, one can see that the solution consists of two symmetric branches $v(\mu) = v(-\mu)$ due to the rotational symmetry of the problem in the complex plane. In other words, the solution is represented by the pair of Cardano equations:

$$v(z\bar{z} + (\pm\mu + \tau v)^2) = \pm\mu + \tau v, \tag{13}$$

since μ, as opposed to r, may be positive or negative. The mapping between r and ξ breaks down when, at some positions $\mu = \pm r_*$, the derivative becomes singular ($d\xi/dr_* = \infty$), as visualized on the left inset at Fig. 1. The set of singular points defines the caustics (sometimes called preshocks). Physically, the singularity comes from the fact that the velocity of the flow is position dependent, which makes the solution, for a given $|z|$, nonunique after a certain time τ. Between the two symmetric caustics (which actually form a conelike surface when viewed from the whole w-complex plane) a shock is formed at $\mu = 0$ for $\tau \geq |z|^2$. Although the shock formation involves the whole (w, z) space, as depicted in Fig. 1, its dynamics is remarkably confined to the region of $r = |w| \to 0$, close to the z plane, which is the basic complex plane in our considerations. As was already mentioned, in this region r plays the role of the regulator ϵ in the formula (1). In this limit the explicit solution of (13) reads

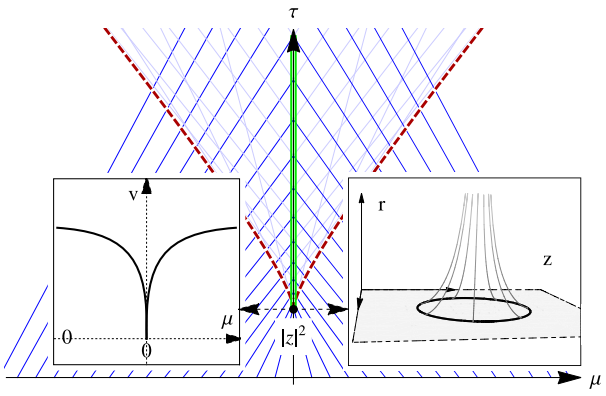

FIG. 1 (color online). The main figure shows, for a given $|z|$, the characteristics (straight lines) and caustics (dashed lines). Inside the later a shock is developed (double vertical line). Left inset shows the solution of Eq. (13) at ($\tau = |z|^2$). Right inset shows the caustics mapped to the $(r = |w|, z)$ hyperplane at the same moment of time. The section $r = 0$ yields the circle $|z|^2 = \tau$, bounding the domain of eigenvalues and eigenvectors correlations for the GE.

$$v^2 = (\tau - |z|^2)/\tau^2 \quad \text{and} \quad v = 0, \quad \text{as r} \to 0. \tag{14}$$

The quantity v^2 has an explicit interpretation [36] in the large N limit, namely,

$$v^2 = \frac{\pi}{N^2}\left\langle \sum_i A_{ii}\delta^2(z - \lambda_i)\right\rangle, \tag{15}$$

where $A_{ij} = (L^\dagger L)_{ij}(R^\dagger R)_{ji}$; i.e., v^2 is a correlator between the biorthogonal sets of left and right eigenvectors introduced before, known in nuclear physics as the Bell-Steinberger matrix [12] and in the RMT context introduced in [8]. This correlator is also known from chaotic scattering theory as the Petermann factor [9]

$$K(z, \tau) = \frac{N}{\pi\rho}v^2 \tag{16}$$

(where ρ is the spectral density calculated later). Off-diagonal elements of matrix A are used to probe nonorthogonality of resonances in open quantum systems [3,37]. Figure 2 shows the time dependence of the Petermann factor for several values of $|z|$. The correlator vanishes outside the critical shock line, where, as we know from the standard approach, the Green's function is analytic, and it is nonzero inside it, where the Green's function is nonanalytic. The edge of the shock line lines up with the contour of the eigenvalue density support. To summarize, the quaternion shock wave dynamics (14) reproduces the result of [8].

Having an explicit solution for v (7), we can turn to g (8). Actually, one can show that g also fulfills a Burgers-like equation exact for any N,

$$\partial_\tau g = v\partial_r g + \frac{1}{N}\Delta_r g, \tag{17}$$

which in the inviscid limit reduces to $\partial_\tau g = v\partial_z g$ or

$$\partial_\tau g = 2v\partial_z v, \tag{18}$$

if one uses $\partial_r g = 2\partial_z v$. We see that we can calculate g by differentiating v. The initial condition $X_0 = 0$ corresponds to $g_0(r) = \bar{z}/(|z|^2 + r^2)$, in particular, $g_0(r = 0) = 1/z$. For $v = 0$ we have $\partial_\tau g = 0$ so g is constant in time, and therefore it is equal to $g = 1/z$ everywhere outside the shock line. Inside the shock line, we employ the second solution of (14), which via elementary integration leads to

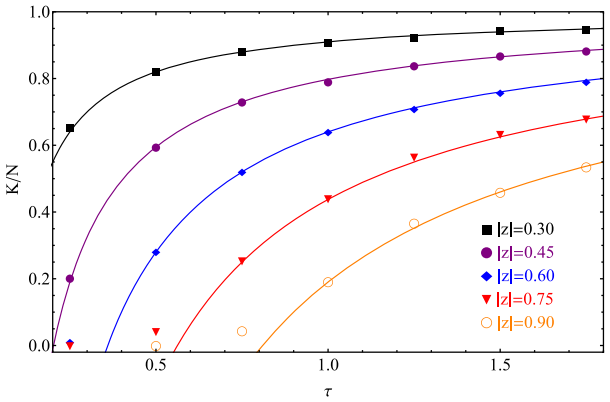

FIG. 2 (color online). The figure depicts theoretical (lines) and numerical (symbols) time dependence of the Petermann factor (rescaled by $1/N$), for different values of $|z|$. For the latter, 3×10^4, 200×200 matrices were used.

$g = \bar{z}/\tau + f(z)$. Since both solutions have to match on the line of the shock due to condition (14), the arbitrary analytic function f has to be equal to zero. Note that for $r = 0$, g coincides with the electric field $G(z, \bar{z})$ in the standard formulation mentioned earlier, so the average spectrum of the considered ensemble reads

$$\rho(z, \tau) = \frac{1}{\pi\tau}\Theta(\tau - |z|^2), \tag{19}$$

where $\Theta(x)$ is the Heaviside step function. We see that complex eigenvalues are uniformly distributed on a growing disc of radius $\sqrt{\tau}$.

Finally, we would like to comment on the solution for large but finite N, at the vicinity of the shock. Since finite size implies nonzero viscosity, the dissipative term will regularize the shock leading to the smoothening of the sharp cliff of the eigenvalue density at the edge of the disk (19). Explicit calculations show that this is indeed the case. The smoothening makes the density at the edge assume a universal shape given by the complementary error function [38]. The argument goes as follows. We use the result of [39], that the spectral density (diagonal part of the kernel) for the Ginibre ensemble is proportional to the $r \to 0$ limit of the characteristic determinant D of the type considered here. The proportionality factor is the normalization C_N and the Gaussian weight $p(z) = \exp(-(N/\tau)|z|^2)$, i.e.,

$$\rho(z, \tau) \overset{N\to\infty}{=} C_N p(z) D(z, r \to 0, \tau), \tag{20}$$

with $C_N = (2/\tau\pi)(1/(N-1)!)(N/\tau)^N$. Then, we may use the fact that the form of D is exactly known for our initial conditions, since it represents the solution for the radial diffusion [27,40,41]

$$D = \int_0^\infty q e^{-N(q^2+r^2/\tau)} I_0\left(\frac{2Nqr}{\tau}\right)(q^2 + |z|^2)^N dq. \tag{21}$$

A careful analysis of the saddle points shows that for large N the main contribution to the integral comes from quantities which scale as $q = \theta N^{-1/4}$, $|z| - \sqrt{\tau} = \eta N^{-1/2}$, and $r = \omega N^{-3/4}$, for θ, η, and ω of order 1. We postpone the details for a future publication. Here we note, however, that this scaling is identical to the critical scaling for the cusp singularity of the Wishart or chiral random matrices. The reason for this lies in the functional form of the determinants, which happens to be identical for the two ensembles. In this way we establish additionally a somehow unexpected link between the universal scaling behavior for the Wishart and Ginibre ensembles. Taking first the large N limit and then setting $\omega = 0$, we recover from (21) a well-known result for the universal scaling at the spectral edge of the Ginibre ensemble

$$\rho(\eta) \approx \frac{1}{2\pi\tau}\mathrm{Erfc}\left(\sqrt{\frac{2}{\tau}}\eta\right). \tag{22}$$

We conclude this Letter with several remarks. First, it is inspiring to compare the Burgers-like structures even between the simplest Hermitian model (GUE) and its non-Hermitian counterpart, i.e., the Ginibre ensemble. In the case of GUE, the characteristic determinant $D_{\mathrm{GUE}}(z)$ fulfills a complex diffusion equation $\partial_\tau D_{\mathrm{GUE}} = -(1/2N)\partial_{zz} D_{\mathrm{GUE}}$. The corresponding Burgers equation resulting from the Cole-Hopf transformation is complex too and has to be solved with complex characteristics. Singularities (shock waves) appear at discrete points (end points of the spectra) in the flow of eigenvalues [21]. On the contrary, for the GE, singularities are given by one-dimensional curves appearing in the flow of eigenvector correlations. The fact that in the Hermitian case the viscosity is negative also has far-reaching consequences. In particular, it is not smoothening the shock, like in the GE (where we observe the Erfc smearing), but it triggers violent oscillations, being the source of Airy universality. Similar universal oscillations originate from negative viscosity in other ensembles. The fact that ensembles as different as GUE, CUE, LUE, and GE have a similar underlying mathematical structure of Burgers-like equations is remarkable and deserves further studies.

Moreover, for clarity we have only considered the dynamics of the simplest non-Hermitian ensemble. Our approach works, however, for any initial condition imposed on the considered process. Additionally, the method can be used to study other non-Hermitian ensembles (e.g., non-Gaussian ones), for which the described coevolution will also be present. The resulting equations are of course much more involved in more general scenarios. Our formalism could also be exploited to expand the area of application of non-Hermitian random matrix ensembles within problems of growth [18], charged droplets in the quantum Hall effect [42], and gauge theory or geometry relations in string theory [43] beyond the subclass of complex matrices represented by normal matrices.

Finally, we would like to emphasize that a consistent description of non-Hermitian ensembles requires the knowledge of the detailed dynamics not only on the complex z plane, where eigenvalues live, but also in the "orthogonal" w plane. In several standard techniques of non-Hermitian random matrix models this second variable is treated as an auxiliary parameter, serving as a regulator only. We have shown that it governs, in the large N limit, the evolution of the standard correlator of eigenvectors which is furthermore coupled to the dynamics of the resolvent. Eigenvectors and eigenvalues evolve therefore simultaneously, and this coevolution is probably a common feature of all, also multipoint Green's functions in non-Hermitian random matrix models.

M. A. N. appreciates discussion with Neil O'Connell on non-Hermitian Brownian walks, which triggered the interest in this problem. J. G., M. A. N., and P. W. would like to thank Jean-Paul Blaizot and Bertrand Eynard for fruitful conversations. The authors are grateful to an anonymous reviewer for a generalization of our argument showing a dependence of D_0 on the eigenvector information (6). P. W. is supported by the International Ph.D Projects Programme of the Foundation for Polish Science within the European Regional Development Fund of the European Union, NSF Agreement No. MPD/2009/6 and the ETIUDA Scholarship under the Agreement No. UMO-2013/08/T/ST2/00105 of the National Centre of Science. M. A. N., Z. B., W. T., and J. G. are supported by the Grant No. DEC-2011/02/A/ST1/00119 of the National Centre of Science NCS.

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Nuclear Physics B 897 (2015) 421–447

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Unveiling the significance of eigenvectors in diffusing non-Hermitian matrices by identifying the underlying Burgers dynamics

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Received 29 March 2015; accepted 1 June 2015

Available online 3 June 2015

Editor: Hubert Saleur

Abstract

Following our recent letter [1], we study in detail an entry-wise diffusion of non-hermitian complex matrices. We obtain an exact partial differential equation (valid for any matrix size N and arbitrary initial conditions) for evolution of the averaged extended characteristic polynomial. The logarithm of this polynomial has an interpretation of a potential which generates a Burgers dynamics in quaternionic space. The dynamics of the ensemble in the large N limit is completely determined by the coevolution of the spectral density and a certain eigenvector correlation function. This coevolution is best visible in an electrostatic potential of a quaternionic argument built of two complex variables, the first of which governs standard spectral properties while the second unravels the hidden dynamics of eigenvector correlation function. We obtain general formulas for the spectral density and the eigenvector correlation function for large N and for any initial conditions. We exemplify our studies by solving three examples, and we verify the analytic form of our solutions with numerical simulations.

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http://dx.doi.org/10.1016/j.nuclphysb.2015.06.002

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1. Introduction

The concept of matrices filled with entries subject to the diffusion process was first introduced by Dyson [2] and applied in the context of both Gaussian Unitary Ensemble (GUE) and Circular Unitary Ensemble (CUE). The arising Coulomb gas analogy had a major impact on understanding of random matrices [3]. Today the marriage of stochastic processes and random matrices brings new insights. Examples include the study of determinantal processes [4–6], Loewner diffusion [7] or the fluctuations of non-intersecting interfaces in thermal equilibrium [8].

Recently, several authors [9,10] have approached the diffusion in the GUE from a new perspective. They found a viscid complex Burgers equation for the logarithmic derivative of the averaged characteristic polynomial $f_N(z,\tau) \equiv \frac{1}{N}\partial_z \ln U_N(z,\tau)$ (associated with a Hermitian matrix filled with entries performing Brownian motion in the complex space):

$$\partial_\tau f_N(z,\tau) + f_N(z,\tau)\partial_z f_N(z,\tau) = -\frac{1}{2N}\partial_z^2 f_N(z,\tau), \tag{1}$$

where τ is the diffusion time and z is a complex variable. The role of viscosity is played by the inverse size of the matrix N. In the $N \to \infty$ limit, $f_N(z,\tau)$ becomes the Green's function $G(z,\tau)$ and the partial differential equation becomes inviscid:

$$\partial_\tau G(z,\tau) + G(z,\tau)\partial_z G(z,\tau) = 0. \tag{2}$$

A solution of the latter equation (by the method of characteristics) requires an introduction of shocks, which turn out to coincide with the edges of the spectra. This phenomenon leads to a novel interpretation of known matrix results, since microscopic universal behavior of the spectra emerges as an expansion around the shock wave of the viscous equation. Nontrivial initial conditions give rise to shock collisions which are equivalent to the merging of the spectrum boundaries. In this way, not only Airy but also the Pearcey functions are captured in the same formalism. A similar Burgers equation was also obtained for the Wishart ensemble and chiral GUE yielding a universal scaling associated with the Bessoid function [11]. The equivalent phenomenon appears also at the level of CUE diffusion, providing new insight for order–disorder transition of Wilson loops in Yang–Mills theory [12].

Recently, this program got extended to the realm of matrices with complex eigenvalues [1]. The success of such an extension is a priori surprising, since hermitian and non-hermitian matrix models seem to be hardly comparable. In the former, the hermiticity condition confines the eigenvalues to the real axis. In the latter, there are no constraints, and the eigenvalues spread over the whole complex plane. Furthermore, non-hermitian models develop discontinuities at the spectral density boundaries, a feature observed even for the well-known Ginibre Ensemble [13], for which the spectral density is given by:

$$\rho(z,\bar{z}) = \frac{1}{\pi}\theta(1-|z|), \tag{3}$$

whereas e.g. in the GUE case, the Wigner semicircle is continuous across the spectral edge, and only the derivatives are discontinuous. Other differences arise when one considers the evolution of the diffusing matrices. In the hermitian case, the evolution is determined by the initial eigenvalues only, whereas in the non-hermitian models, the information on initial eigenvectors additionally affects the shape of the spectral density.

This paper is a continuation and an extension of the ideas of non-hermitian diffusion announced briefly in [1].

Our approach is rooted in the standard electrostatic analogy, but requires a novel setting (we call it the quaternionic method), which can be viewed as an extension of the standard Dysonian strategy applied originally to the hermitian (or unitary) matrix diffusion case. The main objects of our interest are the spectral density (obtained from the resolvent) *and* a certain one-point eigenvector correlation function. We stress that the aforementioned eigenvector correlator is crucial for understanding the diffusion process of non-hermitian matrices. We also point out, why the importance of this correlator was disregarded in majority of the studies of non-hermitian random matrix models.

The basic object of our studies is an averaged "extended" characteristic polynomial (AECP). An extension follows from an introduction of two pairs of complex variables (compared to one complex variable in standard treatments). Surprisingly, AECP obeys a certain partial differential equation akin to the diffusion equation, for arbitrary size of the matrix and for arbitrary initial conditions, and is exactly integrable. The diffusion takes place in the auxiliary plane "perpendicular" to the complex plane where the eigenvalues reside. In the large matrix size limit, the logarithm of the AECP can be viewed as an electrostatic potential and its derivatives with respect to the two complex variables yield a pair of coupled Burgers-like equations for the non-hermitian Green's function and eigenvector correlation function. We would like to mention, that in the standard electrostatic analogy [14–16] the "second variable" is treated as an infinitesimal regulator only. This is the reason why the dynamics, as a function of this variable, remained hidden, and the complementary information on the eigenvector correlator co-evolving with the spectra was absent.

To illustrate our findings, we consider a couple of examples of initial conditions. In most of them we demonstrate the explicit solutions of Burgers equations, obtaining formulas for the spectral density, eigenvector correlators and the electrostatic potential. We note that an inspection of the Burgers-like equation identifies the shock line with the non-holomorphic sector of the spectral density. Moreover, we show the insensitivity of the shock formation to the initial condition chosen. This hints to a lack of truly new universality classes in this type of models, which is corroborated by direct studies of the universal behavior in the vicinity of the spectral collision in one of the examples.

The paper is organized as follows. In Section 2 we define Dysonian non-hermitian diffusion. In Section 3 we briefly review the electrostatic analogy and the quaternionic method. We proceed in Section 4 by deriving the partial differential equation for the AECP and presenting its integral representation. In Section 5 we derive a pair of coupled Burgers equations for the diagonal and off-diagonal parts of the quaternionic Green's function (in the large N limit) thus making a link to the quaternionic method. Subsequently we solve them with the method of complex characteristics. Finally, we obtain an implicit solution to the equation for the potential in terms of the Hopf–Lax formula and derive large N formulas for the spectral density, the eigenvector correlation functions and the boundary of the spectrum valid for an arbitrary initial matrix. Section 6 is devoted to the examples of a) Ginibre, b) spiric and c) 1-band non-normality matrices. We apply previously described methods to these cases, depict the characteristics picture and obtain the large N limit spectral density and eigenvector correlators. We also comment on critical behavior of the AECP. Section 7 is devoted to a curious observation by Osada [17], which actually has triggered our interest in the diffusion of the Ginibre ensemble. We provide also an explanation of the Osada observation. Section 8 summarizes the paper and outlines some possibilities of further investigations of the observed patterns.

Three appendices hide technicalities: Appendix A demonstrates the derivation of the key diffusion equation for an AECP, Appendix B clarifies the link between Ginibre and Wishart/chiral

ensembles, and Appendix C determines weights and normalizations needed for establishing the universal scaling at the shock.

2. Dysonian non-hermitian diffusion

Consider a non-hermitian $N \times N$ matrix $X = (X_{ij})_{i=1,\dots,N\ j=1,\dots,N}$ whose elements $X_{ij} = x_{ij} + i y_{ij}$ undergo $2N^2$-dimensional Brownian motion

$$dx_{ij}(\tau) = \frac{1}{\sqrt{2N}} dB^x_{ij}(\tau), \quad dy_{ij}(\tau) = \frac{1}{\sqrt{2N}} dB^y_{ij}(\tau), \tag{4}$$

where B^x_{ij} and B^y_{kl} are independent Wiener processes. We restrict ourselves to deterministic initial conditions, that is we assume that each element of the matrix has a given initial value $x_{ij} = (x_0)_{ij}$ and $y_{ij} = (y_0)_{ij}$ for $\tau = 0$. This condition can be concisely written as $X = X_0$, for $\tau = 0$. Clearly, this model is a straightforward extension of the Dyson random walk [2] to the realm of non-hermitian random matrices.

The joint probability density for matrix elements evolves according to the $2N^2$-dimensional diffusion equation

$$\partial_\tau P(X,\tau) = \frac{1}{4N} \sum_{ij} (\partial^2_{x_{ij}} + \partial^2_{y_{ij}}) P(X,\tau), \tag{5}$$

with the initial condition $P(X,0) = \delta(X - X_0)$. The probability measure for random matrices at time τ is defined by $d\mu_\tau(X) = \mathcal{D}[X] P(X,\tau)$ where $\mathcal{D}[X] = \prod_{ij} dx_{ij} dy_{ij}$, and the statistical averages by

$$\langle F(X) \rangle_\tau = \int \mathcal{D}[X] P(X,\tau) F(X). \tag{6}$$

The hermitian version of the model, discussed by Dyson, reduces to a model of evolution of eigenvalues. In that case eigenvectors can be integrated out. What makes the non-hermitian extension interesting is that in addition to eigenvalues one has to control also the evolution of eigenvectors. We present a systematic method to do so.

3. Electrostatic analogy and quaternions

In this section we briefly recall the method to calculate eigenvalue distribution of random matrices in the limit $N \to \infty$. The method is based on "electrostatic" analogy [14–16]. One defines a quantity

$$\Phi(z,w,\tau) = \frac{1}{N} \left\langle \mathrm{Tr} \log \left((z - X)(\bar{z} - X^\dagger) + |w|^2 \right) \right\rangle_\tau, \tag{7}$$

which can be interpreted in the limit $w \to 0$ as an electrostatic potential of a cloud of N electric charges interacting on the z-complex plane. The corresponding electric field is

$$G(z,w,\tau) = \partial_z \Phi(z,w,\tau) = \frac{1}{N} \left\langle \mathrm{Tr} \frac{\bar{z} - X^\dagger}{(z - X)(\bar{z} - X^\dagger) + |w|^2} \right\rangle_\tau. \tag{8}$$

Identifying the real and imaginary part of as vector components $G = (E_x - i E_y)/2$ one can rewrite the last equation in the vector notation as $\vec{E} = (E_x, E_y) = \vec{\nabla}_z \Phi$. The minus sign in front of E_y and the scale factor $1/2$ in the relation of G to electric field $\vec{E}$ is a matter of convention.

We are interested in the eigenvalue distribution

$$\rho(z,\tau) \equiv \frac{1}{N}\left\langle \sum_i \delta^{(2)}(z-z_i) \right\rangle_\tau , \tag{9}$$

where z_i's are the eigenvalues of X. The limiting eigenvalue density can be calculated from the Gauss law

$$\rho(z,\tau) = \frac{1}{\pi}\partial_{\bar z} G(z,w,\tau) \ , \ w \to 0. \tag{10}$$

This relation follows from a standard representation of the complex Dirac delta function $\pi\delta^{(2)}(z-z_i) = \lim_{|w|\to 0} \frac{|w|^2}{\left(|w|^2+|z-z_i|^2\right)^2}$. The expression in the brackets on the r.h.s. of (8) can be cast into the standard form of resolvent $(z-X)^{-1}$ at the price of introducing $2N\times 2N$ matrices

$$\mathcal{Q} = \begin{pmatrix} z & -\bar w \\ w & \bar z \end{pmatrix} , \quad \mathcal{X} = \begin{pmatrix} X & 0 \\ 0 & X^\dagger \end{pmatrix}, \tag{11}$$

in place of the original $N\times N$ ones. The resolvent is a 2×2 matrix

$$\mathcal{G}(z,w,\tau) \equiv \begin{pmatrix} \mathcal{G}_{11} & \mathcal{G}_{1\bar 1} \\ \mathcal{G}_{\bar 1 1} & \mathcal{G}_{\bar 1\bar 1} \end{pmatrix} = \frac{1}{N}\left\langle \mathrm{bTr}\frac{1}{\mathcal{Q}-\mathcal{X}} \right\rangle_\tau , \tag{12}$$

where the block-trace is defined as

$$\mathrm{bTr}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \mathrm{Tr}A & \mathrm{Tr}B \\ \mathrm{Tr}C & \mathrm{Tr}D \end{pmatrix}.$$

We refer to $\mathcal{G}(z,w,\tau)$ (12) as to generalized Green's function or quaternionic resolvent [18,19]. Note, that we use the representation of the quaternion in terms of Pauli matrices, i.e. $\mathcal{Q} = q_0 1_2 + i\sum_{j=1}^3 \sigma_j q_j$, so $z = q_0 + iq_1$ and $-\bar w = q_2 + iq_3$. The diagonal element of the quaternionic resolvent $\mathcal{G}_{11}$ is equal to $G(z,w,\tau)$ (8). The extended Green's function $\mathcal{G}(z,w)$ is an advantageous object since one can apply geometric series expansion to $(\mathcal{Q}-\mathcal{X})^{-1}$ which has a diagrammatic interpretation. This leads to a closed set of Dyson–Schwinger equations enumerating underlying planar Feynman diagrams. From these equations one can derive an exact form of the Green's function in the limit $N\to\infty$, as well as matrix-valued addition and multiplication laws [20]. We mention that the quaternionic extension is equivalent to another approach known under the name of hermitization method [21–23], in which the diagonal and off-diagonal blocks of matrices $\mathcal{Q}$ and $\mathcal{X}$ are flipped before the block-trace operation.

Having determined the quaternionic resolvent $\mathcal{G}(z,w,\tau)$ one can determine the potential $\Phi(z,w,\tau)$ or vice versa, since the two objects are related by a simple relation:

$$\mathcal{G} = \begin{pmatrix} \partial_z\Phi & \partial_w\Phi \\ -\partial_{\bar w}\Phi & \partial_{\bar z}\Phi \end{pmatrix}. \tag{13}$$

As follows from (10), the eigenvalue density can be derived from the potential $\Phi(z,w,\tau)$ using the Poisson equation

$$\rho(z,\tau) = \frac{1}{\pi}\partial_{\bar z z}\Phi(z,0,\tau). \tag{14}$$

It turns out that the potential $\Phi(z,w,\tau)$ encodes also information about the correlations of eigenvectors [24], being the special case of the Bell–Steinberger matrix [25–27]. One defines the correlation function as[1]

$$O(z,\tau) \equiv \frac{1}{N^2}\left\langle \sum_\alpha O_{\alpha\alpha}\delta^2(z-z_\alpha)\right\rangle_\tau , \tag{15}$$

with $O_{\alpha\beta} = \langle L_\alpha|L_\beta\rangle\langle R_\alpha|R_\beta\rangle$ where $|L_\alpha\rangle$ ($|R_\alpha\rangle$) are the left (right) eigenvectors of matrix X. It can be shown [28] that, in the $N\to\infty$ limit, this correlation function is related to the off-diagonal elements of the resolvent as $O(z,\tau) = -\frac{1}{\pi}\mathcal{G}_{1\bar 1}(z,0,\tau)\mathcal{G}_{\bar 1 1}(z,0,\tau)$. Applying (13) we have

$$O(z,\tau) = \frac{1}{\pi}\,\partial_w\Phi(z,w,\tau)\partial_{\bar w}\Phi(z,w,\tau)|_{w=0} = \frac{1}{\pi}|V(z,0,\tau)|^2, \tag{16}$$

where $V(z,w,\tau) = \partial_w\Phi(z,w,\tau)$ is the velocity field, which plays the same role in the w-complex plane as the electric field $G(z,w,\tau) = \partial_z\Phi(z,w,\tau)$ in the z-complex plane. It is a vector field. If we parametrize positions on the w-complex plane as $w = a + ib$ and $V = (V_a - iV_b)/2$, then $\vec V = \vec\nabla_w\Phi$. The term "velocity" is related to the underlying Burgers dynamics to be discussed later. To summarize, the limiting eigenvalue density and the eigenvector correlation function can be calculated from the electrostatic potential using eqs. (14) and (16), respectively, and taking the limit $w\to 0$ which project quaternions to the z-plane.

It remains to show how to calculate the electrostatic potential $\Phi(z,w,\tau)$ (7). The standard method is based on enumeration of planar diagrams as mentioned above. The main object in this method is the Green's function $\mathcal{G}$.

Here we propose an alternative approach which is based on the diffusion equation in the quaternionic $(2+2)$-dimensional space in the direction perpendicular to the z-complex plane. The primary object in this calculation is an extended characteristic polynomial defined in the next section.

4. Averaged extended characteristic polynomial

In order to calculate the potential (7) in the limit $N\to\infty$ we rewrite it as

$$\Phi(z,w,\tau) = \frac{1}{N}\left\langle \log\det\left((z-X)(\bar z - X^\dagger) + |w|^2\right)\right\rangle_\tau , \tag{17}$$

and define an associated object – an effective potential

$$\phi(z,w,\tau) = \frac{1}{N}\log\left\langle \det\left((z-X)(\bar z - X^\dagger) + |w|^2\right)\right\rangle_\tau \equiv \frac{1}{N}\log D(z,w,\tau), \tag{18}$$

where

$$D(z,w,\tau) = \left\langle \det\left((z-X)(\bar z - X^\dagger) + |w|^2\right)\right\rangle_\tau = \left\langle \det\begin{pmatrix} z-X & -\bar w \\ w & \bar z - X^\dagger\end{pmatrix}\right\rangle_\tau . \tag{19}$$

We refer to $D(z,w,\tau)$ as to averaged extended characteristic polynomial (AECP). For the Gaussian process (4) the determinant $D(z,w,\tau)$ self-averages for $N\to\infty$ and Φ can be replaced by

[1] Note that we introduced an additional $1/N$ factor as compared to the definition given in [24] to obtain a limiting density.

ϕ in this limit. The advantage of using the latter is that the averaged extended characteristic polynomial $D(z, w, \tau)$ (19), which appears in the definition of $\phi = \frac{1}{N} \log D$ obeys a simple diffusion equation with respect to the variable w

$$\partial_\tau D(z, w, \tau) = \frac{1}{N} \partial_{w\bar{w}} D(z, w, \tau), \tag{20}$$

as shown in Appendix A. Note that, from the point of view of this equation, z is a dummy parameter. The z-dependence appears only in the initial condition

$$D(z, w, 0) = D_0(z, w) = \det\left((z - X_0)(\bar{z} - X_0^\dagger) + |w|^2\right), \tag{21}$$

which is completely determined by the initial matrix X_0. In other words, for each z we have an independent diffusion in the perpendicular w-complex plane. The problem is therefore exactly integrable, and the solution of the diffusion equation (20) reads

$$D(z, w, \tau) = \frac{N}{\pi\tau} \int_C \exp\left(-N \frac{|w - w'|^2}{\tau}\right) D_0(z, w') \mathrm{d}^2 w', \tag{22}$$

where $D_0(z, w') = D(z, w', 0)$. The solution can be equivalently written as

$$\phi(z, w, \tau) = \frac{1}{N} \log \frac{N}{\pi\tau} \int_C \exp N\left(\phi_0(z, w') - \frac{|w - w'|^2}{\tau}\right) \mathrm{d}^2 w', \tag{23}$$

where $\phi_0(z, w') = \frac{1}{N} \log D_0(z, w')$. In the limit $N \to \infty$ the last equation assumes the form of the Hopf–Lax formula [29]

$$\phi(z, w, \tau) = \max_{w'} \left(\phi_0(z, w') - \frac{|w - w'|^2}{\tau}\right). \tag{24}$$

This equation describes the evolution of the electrostatic potential in the limit $N \to \infty$ for the given initial configuration $\phi_0(z, w)$, so this equation solves our original problem.

A few remarks are in order. The characteristic polynomial $D(z, w, \tau)$ and the potential $\phi(z, w, \tau)$ (18) depend on w only through the norm $|w|^2$. The diffusion preserves the spherical symmetry of these quantities in the w-complex plane, so it is convenient to rewrite these equations in the radial part r of $w = re^{i\alpha}$ skipping the dependence on the phase α. In particular (22) takes the form

$$D(z, r, \tau) = \frac{2N}{\tau} \int_0^\infty r' \exp\left(-N \frac{r^2 + r'^2}{\tau}\right) I_0\left(\frac{2Nrr'}{\tau}\right) D_0(z, r') \mathrm{d}r', \tag{25}$$

and (24) simplifies to

$$\phi(z, r, \tau) = \max_{r'} \left(\phi_0(z, r') - \frac{(r - r')^2}{\tau}\right). \tag{26}$$

The second remark is on the role of the parameter w. Originally it was introduced as a regulator to the expression for the potential (7) and eventually sent to zero. Here we promote w to a full complex variable and analyse the dynamics of the model on the entire w-complex plane. This approach allows one to trace not only eigenvalues but also eigenvectors of the random

matrix X and to break the symmetry between matrices having identical eigenvalues but different eigenvectors.

A complex valued matrix can be Schur decomposed $X = U(\Lambda + T)U^\dagger$ where U is a unitary matrix, Λ is a diagonal matrix containing the complex eigenvalues, and T is a strictly upper-triangular matrix encoding information about eigenvectors. Two different matrices X_1, X_2 having the same eigenvalues Λ but different eigenvectors have different T_1 and T_2. The averaged extended characteristic polynomial (19) for these matrices differs $D_1(z, w) \neq D_2(z, w)$ when $|w| \neq 0$. From this difference one can read off information about eigenvectors.

Moreover, the dynamics of the model in the quaternionic space has a beautiful physical interpretation in terms of the Burgers dynamics. The behavior of the model on the z-complex plane is a shadow of this dynamics. In particular, the support of the eigenvalue density $\rho(z, \tau)$ coincides with the location of shocks of the quaternionic Burgers dynamics in the full quaternionic (z, w)-space.

5. Burgers dynamics

In the previous section we have found a solution to the diffusion equation (20). Here for completeness we relate the diffusion equation to Burgers dynamics. Using the definition (18) it is easy to see that the effective potential $\phi = \phi(z, w, \tau)$ and its gradient $v(z, w, \tau) = \partial_w \phi(z, w, \tau)$ fulfill the following differential equations

$$\partial_\tau \phi = \frac{1}{N} \partial_{w\bar{w}} \phi + \partial_w \phi \partial_{\bar{w}} \phi, \tag{27}$$

and

$$\partial_\tau v = \frac{1}{N} \partial_{w\bar{w}} v + \partial_w |v|^2, \tag{28}$$

respectively. These equations describe Burgers dynamics on the w-complex plane for a two dimensional velocity field $v = (V_a - i V_b)/2$ derived from the potential ϕ: $\vec{V} = \vec{\nabla}_w \phi = (\partial_a \phi, \partial_b \phi)$ where $w = a + ib$. The coefficients of the Laplacian term can be identified as a hydrodynamic viscosity parameter $\nu = \frac{1}{N}$. Equation (28) follows from (20) by an inverse Cole–Hopf transformation [30]. One can also write an equation for the z-gradient, $g(z, w, \tau) = \partial_z \phi(z, w, \tau)$:

$$\partial_\tau g = \frac{1}{N} \partial_{w\bar{w}} g + \partial_z |v|^2. \tag{29}$$

The two gradients are related to each other as $\partial_w g = \partial_z v$. The effective potential ϕ and the gradients reproduce the electrostatic potential and the quaternionic Green's function in the limit $N \to \infty$

$$\phi \longrightarrow \Phi, \qquad \begin{pmatrix} g & v \\ -\bar{v} & \bar{g} \end{pmatrix} \longrightarrow \mathcal{G}. \tag{30}$$

Let us now discuss the inviscid limit $N \to \infty$. The effective potential (27) obeys the equation

$$\partial_\tau \phi = \partial_w \phi \partial_{\bar{w}} \phi, \tag{31}$$

which after applying rotational symmetry (in the variable w) simplifies to an equation

$$\partial_\tau \phi = \frac{1}{4} (\partial_r \phi)^2, \tag{32}$$

for the radial variable $r = |w|$. The solution is given by the Hopf–Lax formula (26) which in our case is equivalent to

$$\phi(z, r, \tau) = \phi_0(z, r_*) - \frac{(r - r_*)^2}{\tau}, \tag{33}$$

with r_* being the location $r_* = r'$ of the maximum (26) given by the usual extremum condition

$$\partial_r \phi_0(z, r_*) = \frac{2(r_* - r)}{\tau}. \tag{34}$$

To complete the scheme, the maximizing parameter r_* has to be calculated from (34) and the result $r_* = r_*(z, r, \tau)$ has to be inserted to (33).

We solve this equation for $\phi_0(z, r) = \frac{1}{N}\mathrm{Tr}\log \mathcal{M}(z, r)$ where $\mathcal{M}(z, r) = (z - X_0)(\bar{z} - X_0^\dagger) + r^2$ with an initial matrix X_0

$$\phi(z, r, \tau) = \frac{1}{N}\mathrm{Tr}\log \mathcal{M}(z, r_*) - \frac{(r - r_*)^2}{\tau}, \qquad \frac{r_*}{N}\mathrm{Tr}\mathcal{M}(z, r_*)^{-1} = \frac{r_* - r}{\tau}. \tag{35}$$

Eliminating r_* from this set of equation we obtain the effective potential $\phi(z, r, \tau)$ for an arbitrary initial matrix X_0. From it we derive the limiting density $\rho(z, \tau)$ (14) and the eigenvector correlations $O(z, \tau)$ (16)

$$\rho(z, \tau) = \frac{1}{\pi}\partial_{z\bar{z}}\phi(z, 0, \tau), \tag{36}$$

$$O(z, \tau) = \frac{1}{4\pi}\lim_{r\to 0}\left(\partial_r \phi(z, r, \tau)\right)^2. \tag{37}$$

For the initial condition of the form $\phi_0(z, r) = \frac{1}{N}\mathrm{Tr}\log \mathcal{M}(z, r)$ we arrive, after differentiation and some algebraic manipulations, at

$$\begin{aligned}\rho(z, \tau) = {} & \frac{1}{N\pi}\frac{1}{\mathrm{Tr}\mathcal{M}^{-2}}\det\begin{pmatrix}\mathrm{Tr}(\bar{z} - X_0^\dagger)\mathcal{M}^{-2} & \mathrm{Tr}\mathcal{M}^{-2}r_* \\ -\mathrm{Tr}\mathcal{M}^{-2}r_* & \mathrm{Tr}(z - X_0)\mathcal{M}^{-2}\end{pmatrix} \\ & + \frac{1}{N\pi}\mathrm{Tr}\left(\mathcal{M}^{-1}[\mathcal{M}^{-1}; z - X_0](\bar{z} - X_0^\dagger)\right),\end{aligned} \tag{38}$$

$$O(z, \tau) = \frac{1}{\pi\tau^2}r_*^2, \tag{39}$$

where $\mathcal{M} = \mathcal{M}(z, r_*)$. In the final formulas we set $r = 0$ to project the results to the z-complex plane. The equation for r_* (35) simplifies for $r = 0$ to

$$\frac{1}{N}\mathrm{Tr}\mathcal{M}(z, r_*)^{-1} = \frac{1}{\tau}. \tag{40}$$

Equations (38), (39) are valid inside the boundary given by

$$\frac{1}{N}\mathrm{Tr}\mathcal{M}(z, 0)^{-1} = \frac{1}{\tau}, \tag{41}$$

which corresponds to $r_* = 0$. Outside this boundary $O(z, \tau) = 0$ and $\rho(z, \tau) = 0$. For a normal initial matrix X_0, the second term in the spectral density (38) drops out since $[\mathcal{M}^{-1}, z - X_0] = 0$.

By inspecting the boundary equation (41) one finds a surprising connection to the so-called pseudospectrum [31], a mathematical concept generalizing the notion of the eigenvalue spectrum. Pseudospectrum of a matrix A is defined as a subset σ_A of the complex plane z such that

$$||(z-A)^{-1}|| > \frac{1}{\epsilon}, \tag{42}$$

where the symbol $||\cdot||$ is some arbitrary matrix norm and ϵ is the parameter of the pseudospectrum. In the $\epsilon \to 0$ limit, one recovers the standard eigenvalues as poles of the resolvent $(z-A)^{-1}$.

For the initial matrix $A = X_0$, the boundary of the pseudospectrum subset σ_{X_0} is exactly the eigenvalue boundary (41) with $\epsilon^2 = \frac{\tau}{N}$ and a Frobenius norm $||X||_F = \sqrt{\mathrm{Tr} X^\dagger X}$. From this simple identification we conclude that a diffusion model with an initial matrix X_0 is also a probabilistic realization of the pseudospectrum for the same matrix.

We finish this section by discussing an equation for the gradient $v = \partial_w \phi$ in the inviscid limit. The equation is equivalent to the one for the potential ϕ that we discussed above but in some situations the equation for the gradient is more handy to use. The inviscid version of (28) reads

$$\partial_\tau v = \partial_w |v|^2. \tag{43}$$

It is an inviscid Burgers equation in $2+1$ dimensions. A general solution to this equation for smooth (differentiable) initial conditions can be deduced from the Hopf–Lax formula (24) for the effective potential ϕ for $N \to \infty$. The maximum in (24) is achieved for $w' = w_*$ fulfilling the standard extremum condition

$$0 = \bar{v}_0(z, w_*) - \frac{w_* - w}{\tau}, \tag{44}$$

which is equivalent to

$$w = w_* - \tau \bar{v}_0(z, w_*), \tag{45}$$

for which $v(z, w, \tau) = v_0(z, w_*)$. Inserting w_* (45) to this equation we get a solution to (43)

$$v = v_0(z, w + \tau \bar{v}), \tag{46}$$

which is given by an implicit equation for $v = v(z, w, \tau)$ depending only on the initial condition $v_0(z, w)$. The parameter w_* can be viewed as a labeling parameter for the family of the characteristic lines. These lines start to cross when the labeling fails to be bijective i.e. $\frac{dw}{dw_*} = 0$. This singularity condition defines a caustic surface – a boundary across which an ambiguity of the solutions arises. The development of multivalued solutions is an unwanted feature of the inviscid Burgers equation and is circumvented by constructing shock lines along which one cuts the characteristics rendering the solution unique.

The two-dimensional Burger evolution (43) can be simplified in our case due to the rotational symmetry. We are interested only in the solutions which depend on the modulus $r = |w|$. In this case the vector velocity field is a central field $v = \frac{\bar{w}}{r} v$, with $v = |v|$, and the vector equation (43) reduces to a scalar equation for the modulus of the velocity field

$$\partial_\tau v = v \partial_r v, \tag{47}$$

with a solution given by

$$v = v_0(z, r + \tau v). \tag{48}$$

From this solution we can reconstruct the full $2d$-solution: $v(z, w, \tau) = \frac{\bar{w}}{r} v(z, r, \tau)$ with $r = |w|$.

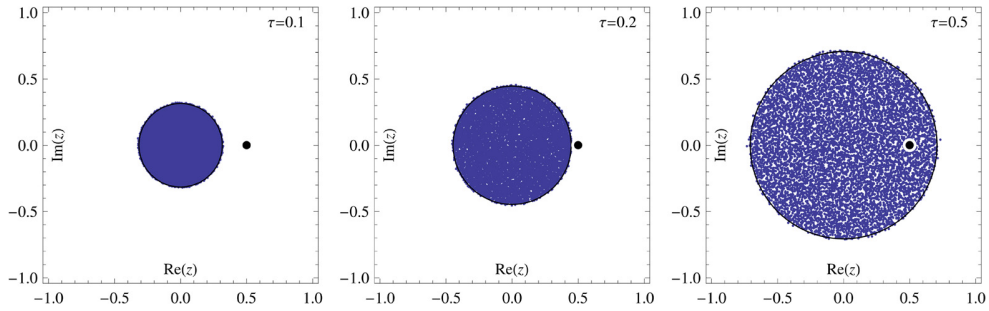

Fig. 1. A numerical simulation of the spectral density with an initial matrix $X_0 = 0$ at time slices $\tau = 0.1$, $\tau = 0.2$ and $\tau = 0.5$ respectively, an ensemble of 6 matrices of size $N = 1500$. Black curves are the large N spectral boundaries and the dot indicates an arbitrary spectator position z where the evolution is probed.

6. Examples

In this section we discuss three examples: (1) the canonical Ginibre evolution for which the initial matrix is $X_0 = 0$, (2) the spiric evolution for which $X_0 = \mathrm{diag}(-a, -a \ldots, a, a \ldots)$ with an equal number of $\pm a$ and (3) an evolution initiated from a non-normal matrix: $(X_0)_{ij} = \alpha\delta_{i,j-1}$. This matrix has eigenvalues equal zero as the initial matrix in the first example but it is not a normal matrix.

The first example serves as a proof-of-concept. We solve the Burgers equations by the method of characteristics to obtain the spectral density, the eigenvector correlator and the potential function in the large N limit. The discussion is accompanied by Appendices B, C where we consider finite w results and a relation of the averaged extended characteristic polynomial to the two-point kernel of the underlying determinantal process.

The second example illustrates an evolution of the eigenvalue density initiated from two disconnected eigenvalue "islands" which grow in the course of time to collide at some critical time. We discuss a novel universality arising in the vicinity of the collision.

The third example demonstrates the dependence of the evolution of the eigenvalue distribution on the initial information which goes beyond the eigenvalues themselves.

6.1. Ginibre evolution

The evolution is initiated from the matrix $X_0 = 0$. The evolution of the eigenvalue density is shown in Fig. 1. The spectral density forms a circular eigenvalue "island" expanding in time. For each time τ the ensemble of matrices in this evolution is equivalent to a Ginibre Ensemble with a time-rescaled dispersion. The initial condition for the determinant (21) is $D_0(z, w) = \left(|z|^2 + |w|^2\right)^N$, for the effective potential (18) $\phi_0(z, w) = \log(|z|^2 + |w|^2)$, for the velocity $v_0(z, w) = \bar{w}/(|z|^2 + |w|^2)$, and for its modulus $v_0(z, r) = r/(|z|^2 + r^2)$, where $r = |w|$, respectively. We solve the inviscid $2 + 1$ Burgers equation (43) using the method of characteristics. Characteristics (45) for this equation are shown in Fig. 2. In the left panel we show a plot in $2 + 1$ dimensions and in the right one its $1 + 1$ dimensional section. The evolution takes place in the w complex plane however the position on z-plane defines the initial condition. The z-variable acts as a spectator on the eigenvalue plane "observing" the evolution in the perpendicular w-direction. We identify the cone-like caustic surface (left plot) whose apex is located at $r = 0$ and at critical time $\tau_c = |z|^2$. This surface is the boundary along which the characteristics

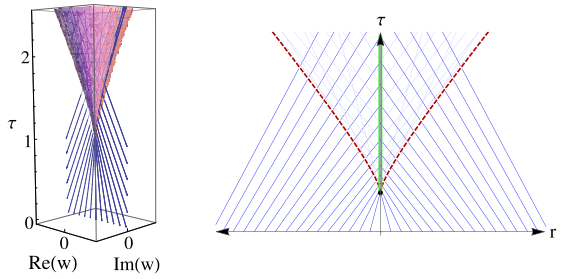

Fig. 2. The characteristic lines at $z = 1$ for the vector $2+1$ Burgers equation (left) and a $1+1$ section (right). The caustic cone-like surface on the left plot is denoted by dashed line on the right one. The shock is located on the cones' axis, it starts from its apex. It is shown a double line on the vertical axis in the right chart. The vertical line corresponds to $r = 0$, that is to the place where the quaternionic pair (z, w) reduces to $(z, 0)$ that lies on the z-complex plane.

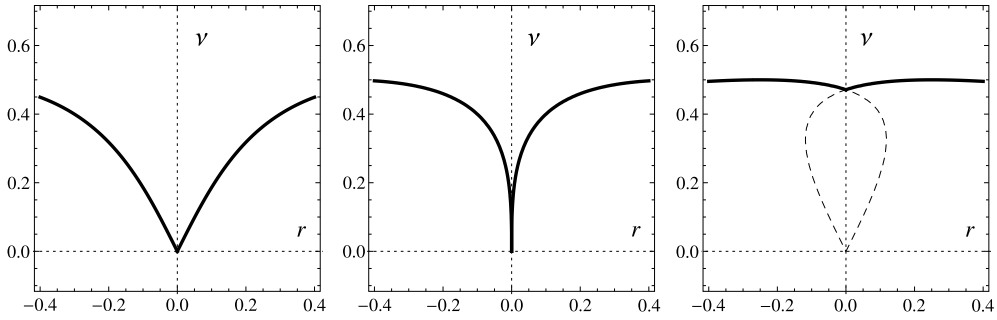

Fig. 3. The solution $|v_\tau|$ for fixed $z_0 = 1$ and three different times $\tau = 0.5$, $\tau = 1$ and $\tau = 1.5$ respectively. It shows the development of nonzero solution for $\tau > \tau_c$ and the emergence of an additional unphysical solution depicted by dashed lines.

start to cross, making the Burgers solution multi-valued. This ambiguity develops for $\tau > \tau_c$ as the expanding eigenvalue boundary "swallows" the spectator at z_0. We find the position of the shock line as a locus distanced equally from the caustic surface at each given time (the Rankine–Hugoniot condition). Since our problem is radially symmetric, the shock is positioned exactly at $r = 0$ and starts from the critical time τ_c. Therefore although the dynamics takes place in the whole r space, the shock is always confined to the "physical" $r = 0$ region. Moreover if we confine it to $r = 0$, the spectator at z stays on the shock line at every time $\tau > \tau_c$. We conclude that inside the bulk of the spectrum (i.e. the non-holomorphic sector), the observer is constantly on the shock line.

Having identified the positions of the shock we can write down a solution of (48) for the reduced $1+1$ Burgers equation for our initial conditions $v_0(z, r) = r/(|z|^2 + r^2)$. It reads

$$v = \frac{r + \tau v}{(r + \tau v)^2 + |z|^2}. \tag{49}$$

It is an implicit algebraic equation for $v = v(z, r, \tau)$. It can be rewritten as a cubic equation. Solutions for different τ are plotted in Fig. 3. Rather than showing the solution for the modulus v we show a cross section of the vector field which has two symmetric branches $\pm v$. The right plot shows the solution inside the caustic surface with an unphysical branch depicted as dashed lines. For $r = 0$, that is on the z-plane, we have

$$v(z,\tau) = \begin{cases} 0 & \text{for } \tau < |z|^2 \\ \frac{1}{\tau}\sqrt{\tau - |z|^2} & \text{for } \tau > |z|^2 \end{cases}. \tag{50}$$

The boundary $|z|^2 = \tau$ is found by the sewing condition of zero and non-zero solutions. From v we readily obtain the eigenvector correlation function (15) in the large N limit

$$O(z,\tau) = \frac{1}{\pi} v^2 = \begin{cases} 0 & \text{for } \tau < |z|^2 \\ \frac{1}{\tau^2 \pi}(\tau - |z|^2) & \text{for } \tau > |z|^2 \end{cases}. \tag{51}$$

We can also find the second gradient $g = \partial_z \phi$ of the effective potential. To this end we can use equation (29) which in the limit $N \to \infty$ simplifies to $\partial_\tau g = \partial_w |v|^2$. For our initial condition $g_0(z,r) = \bar{z}/(|z|^2 + r^2)$ it gives

$$g(z,\tau) = \begin{cases} 1/z & \text{for } \tau < |z|^2 \\ \bar{z}/\tau & \text{for } \tau > |z|^2 \end{cases}. \tag{52}$$

Now we can use the Gauss law (10) to obtain the limiting eigenvalue density

$$\rho(z,\tau) = \frac{1}{\pi\tau}\theta(\sqrt{\tau} - |z|). \tag{53}$$

It is given by a uniform distribution on a disc of radius $\sqrt{\tau}$. For $\tau = 1$, results (51), (53) reproduce results of [13,24], respectively. We could alternatively find the same formulas by directly applying (38) and (39). The maximizer r_* can be found by solving the constraint (40):

$$r_* = \begin{cases} 0 & \text{for } \tau < |z|^2 \\ \sqrt{\tau - |z|^2} & \text{for } \tau > |z|^2 \end{cases}. \tag{54}$$

We find

$$\phi(z,\tau) = \begin{cases} \ln|z|^2 & \text{for } \tau < |z|^2 \\ \ln\tau + \frac{|z|^2}{\tau} - 1 & \text{for } \tau > |z|^2 \end{cases}. \tag{55}$$

The last formula was obtained for $\tau = 1$ in [32].

So far we have discussed the limit $N \to \infty$, in which as the effective potential ϕ and its gradients are equal to the electrostatic potential and the quaternionic Green's function (30). What about finite N? For finite N the order of calculating the average and the logarithm in (17) and (18) matters, so ϕ is at best only an approximation of Φ for large but finite N. So clearly our method is not able to get insight into the finite N corrections. Surprisingly as we show below, the behavior of the diffusion kernel $D(z,w,\tau)$ on the z complex plane, that is for $w = 0$ reveals the same type of finite size effects as known from exact calculations of the eigenvalue density for Ginibre ensemble [24]. For our initial conditions the diffusion kernel (25) is

$$D(z,r,\tau) = \frac{2N}{\tau}\int_0^\infty r' \exp\left(-N\frac{r^2 + r'^2}{\tau}\right) I_0\left(\frac{2Nrr'}{\tau}\right)(|z|^2 + r'^2)^N \mathrm{d}r' \tag{56}$$

On the z-complex plane that is for $r = 0$ this integral significantly simplifies and it can be for large N calculated using the saddle point method. There are three saddle points

$$r'_0 = 0, r'_\pm = \pm\sqrt{\tau - |z|^2}. \tag{57}$$

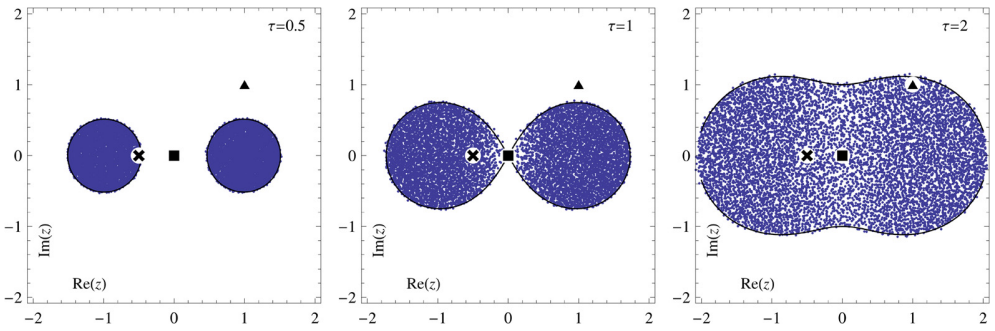

Fig. 4. The spectral density dynamics for an initial matrix $X_0 = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$ before, at and after the critical time $\tau_c = 1$, the ensemble consisted of 6 matrices of size $N = 1500$. The cross, square and triangle denote three observers useful in the analysis of the evolution.

When τ approaches $|z|^2$ from above, the two points $r'_\pm$ approach each other along the real axis to coalesce for $\tau = |z|^2$ at $r' = 0$. When τ becomes smaller than $|z|^2$ the two points move on the imaginary axis. Near the critical value $\tau = |z|^2$ it is convenient to introduce rescaled parameters

$$r' = \theta N^{-1/4}, \qquad |z| = \sqrt{\tau} + \eta N^{-1/2} \tag{58}$$

and use them in the calculations. One finds that for large N the diffusion kernel behaves as

$$D(z = \sqrt{\tau} + \eta N^{-1/2}, r = 0, \tau) \sim \frac{1}{2\pi\tau} \mathrm{erfc}\left(\sqrt{\frac{2}{\tau}}\eta\right), \tag{59}$$

where erfc is the complementary error function. This result holds not only on the z-complex plane but also sufficiently close to the z-complex plane that is for r approaching zero as $N^{-3/4}$ or faster: $r = O(N^{-3/4})$. Indeed in this case the argument of the function I_0 is a finite number and this function behaves as a constant for large N. The error function behavior has the same form as the finite size expression for the eigenvalue density function. This indicates that there may be some deeper relation between the diffusion kernel (averaged extended characteristic polynomial) and the two-point kernel known from the considerations of the underlying determinantal process for Ginibre matrices. We work out this analogy on a heuristic level in Appendices B, C.

6.2. Spiric case

We now consider diffusion initiated from a diagonal matrix $X_0 = \mathrm{diag}(a, \ldots, a, -a, \ldots, -a)$ with the same number of a and $-a$ eigenvalues. In the course of time two initial eigenvalue "islands", initially concentrated around $\pm a$, expand to collide at some critical time τ_c as shown in Fig. 4. The initial condition corresponds to

$$D_0(z, r) = (r^2 + |z - a|^2)^{\frac{N}{2}} (r^2 + |z + a|^2)^{\frac{N}{2}},$$

or if we write it for the modulus of the velocity field (47)

$$v_0(z, r) = \frac{1}{2}\frac{r}{r^2 + |z - a|^2} + \frac{1}{2}\frac{r}{r^2 + |z + a|^2}. \tag{60}$$

In Fig. 5 we show characteristics for three different values of the observer position z, including $z = 0$ where the collision takes place. As we can see the dynamics in the r plane behaves qualitatively in the same way as for the Ginibre case. In particular, characteristics form the same type

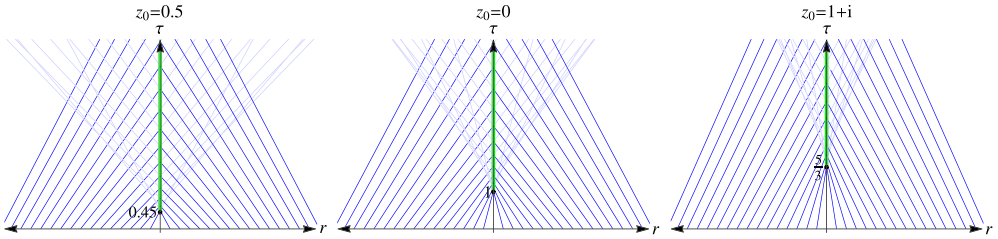

Fig. 5. Characteristic lines in the spiric case for fixed observers $z_0 = -0.5$ (a cross on Fig. 4), $z_0 = 0$ (a square on Fig. 4) and $z_0 = 1 + i$ (a triangle on Fig. 4) with $a = 1$. From left to right, the shock formation occurs later in time which corresponds to meetings between observers z_0 and the expanding spectral boundary.

of caustic surfaces with a shock line present after some critical time $\tau_c = \frac{|a^2 - z^2|^2}{|a|^2 + |z|^2}$. This critical time can be found by sewing the solutions ν or by finding the position of the caustic surface apex as a function of z and τ. In Fig. 6 we show a comparison of the caustic surfaces for Ginibre and spiric evolution for $\tau = 0.5$. For this initial condition the solution (48) of the inviscid Burgers equation takes the form

$$2\nu = \frac{r + \tau\nu}{|z + a|^2 + (r + \tau\nu)^2} + \frac{r + \tau\nu}{|z - a|^2 + (r + \tau\nu)^2}. \tag{61}$$

For $r = 0$ the solution can be written explicitly

$$\nu(z, r = 0; a) = \begin{cases} 0 & \text{for } z \notin \mathcal{S} \\ \frac{1}{\tau}\sqrt{\frac{\tau}{2} - |a|^2 - |z|^2 + \frac{1}{2}S_a} & \text{for } z \in \mathcal{S} \end{cases}, \tag{62}$$

where $S_a = \sqrt{\tau^2 + 4Z_a^2}$, $Z_a = \bar{z}a + z\bar{a}$. The symbol $\mathcal{S}$ stands for the interior of the spiric section defined by the contour

$$\tau(|a|^2 + |z|^2) = |a^2 - z^2|^2. \tag{63}$$

Generally spiric section is a curve obtained by intersection of a torus and a plane parallel to its rotational symmetry axis. We plotted spiric curves as contours around the scatter plots for eigenvalue densities in Fig. 4. The eigenvector correlation function inside the spiric section $\mathcal{S}$ is

$$O(z, \tau) = \frac{\nu^2}{\pi} = \frac{N}{\tau^2\pi}\left(\frac{\tau}{2} - |a|^2 - |z|^2 + \frac{1}{2}S_a\right). \tag{64}$$

We can also calculate the diagonal element of the Green's function using equation (29) that in the limit $N \to \infty$ simplifies to $\partial_\tau g = \partial_z \nu^2$. For the initial condition as $g_0 = \frac{1}{2}\left(\frac{\bar{z}+\bar{a}}{r^2+|z+a|^2} + \frac{\bar{z}-\bar{a}}{r^2+|z-a|^2}\right)$ the solution reads

$$g(z, r = 0, \tau) = \begin{cases} \frac{z}{z^2 - a^2} & \text{for} \quad z \notin \mathcal{S} \\ \frac{\bar{z}}{\tau} - \frac{\bar{a}S_a}{2\tau Z_a} + c(z, a) & \text{for} \quad z \in \mathcal{S} \end{cases}, \tag{65}$$

with a constant $c(z, a) = \frac{\bar{a}}{2Z_a}$ obtained by the sewing condition along the spiric section $\mathcal{S}$. We use the Gauss law to obtain the spectral density as

$$\rho(z, \tau) = \frac{S_a(2 - \tau|a|^2) + \tau^2|a|^2}{2\pi\tau Z_a^2 S_a}. \tag{66}$$

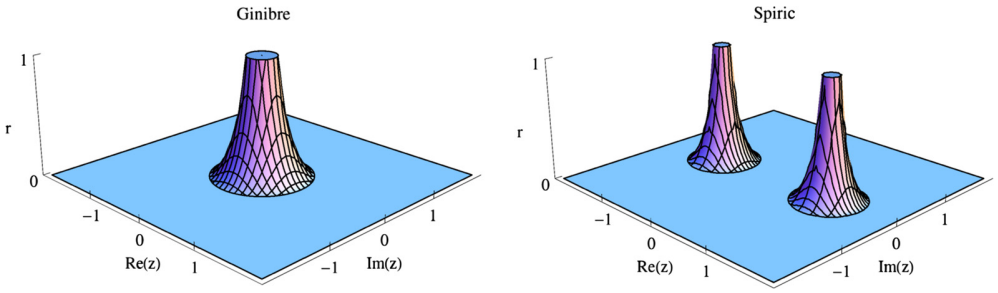

Fig. 6. Caustic surface in the variables (r, z) for Ginibre case (left plot) and spiric case (right plot) with time fixed at $\tau_0 = 0.5$.

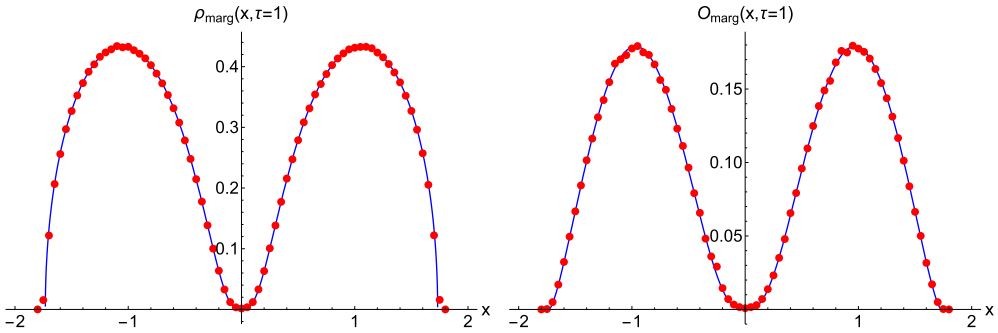

Fig. 7. Numerical results in the spiric case for $a = 1$ and critical time $\tau = 1$ for eigenvector correlator (right plot) and spectral density (left plot) averaged over the imaginary axis, the ensemble consisted of $6 \cdot 10^3$ matrices of size $N = 1000$.

The same results can be obtained from the calculations of the effective potential ϕ using equations (38), (39) and (40) for r_*:

$$r_* = \begin{cases} 0 & \text{for } z \notin \mathcal{S} \\ \sqrt{\frac{\tau}{2} - |a|^2 - |z|^2 + \frac{1}{2}S_a} & \text{for } z \in \mathcal{S} \end{cases}. \tag{67}$$

Outside $\mathcal{S}$ we get

$$\phi^{out}(z,\tau) = \frac{1}{2}\ln|z-a|^2 + \frac{1}{2}\ln|z+a|^2, \tag{68}$$

and inside $\mathcal{S}$

$$\phi^{in}(z,\tau) = \frac{1}{2}\ln\left(\frac{\tau^2}{2} + \frac{\tau}{2}S_a\right) + \frac{1}{\tau}\left(|a|^2 + |z|^2 - \frac{\tau}{2} - \frac{1}{2}S_a\right). \tag{69}$$

We recall that $S_a = \sqrt{\tau^2 + 4Z_a^2}$, $Z_a = \bar{z}a + z\bar{a}$, as defined after (62). In Fig. 7 we show a comparison of theoretical predictions for the limiting eigenvalue density (66) and the eigenvector correlation function (64) with Monte-Carlo simulations. The agreement is very good.

We complete the discussion of the spiric case by deriving a finite N formula for the diffusion kernel $D(z, r, \tau)$

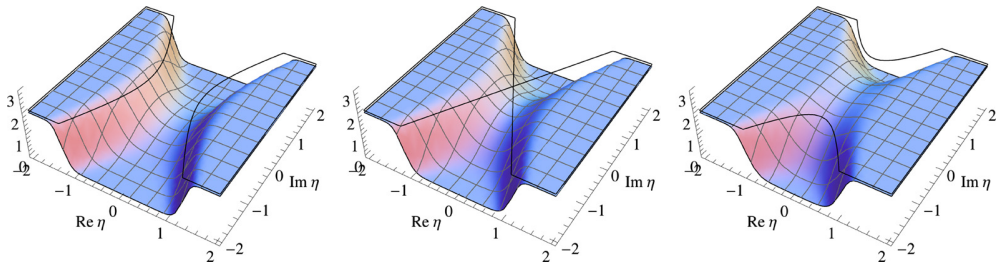

Fig. 8. Evolution of the characteristic polynomial in the vicinity of the collision for rescaled times $t=-1$, $t=0$, $t=1$. Black contour on top is the large N boundary of non-zero spectral density obtained from (63).

$$D(z,r,\tau)=\int_0^\infty r'\exp\left(-N\frac{r^2+r'^2}{\tau}\right)I_0\left(\frac{2Nrr'}{\tau}\right)(r'^2+|z-a|^2)^{\frac{N}{2}}(r'^2+|z+a|^2)^{\frac{N}{2}}\,\mathrm{d}r'. \tag{70}$$

We are interested in the behavior close to the origin $z=0$, $r=0$ for τ close to the collision time. Without loss of generality we perform calculations for $a=1$. In this case the collision time is $\tau=1$. We set $r=0$ and zoom into the vicinity of the critical region

$$r'=\theta N^{-1/4},\qquad z=\eta N^{-1/4},\qquad \tau=1+tN^{-1/2} \tag{71}$$

where the saddle points merge. After expanding the solution (70) we obtain an asymptotic formula

$$\begin{aligned}&D(z=\eta N^{-1/4},r=0,\tau=1+tN^{-1/2})\\&\quad\sim\sqrt{\frac{\pi}{128N}}e^{-\frac{\sqrt{N}}{2}((\eta+\bar\eta)^2-2|\eta|^2)}(\eta+\bar\eta)^4\operatorname{erfc}\left[\frac{1}{\sqrt{2}}\left(|\eta|^2-(\eta+\bar\eta)^2-t\right)\right],\end{aligned} \tag{72}$$

which is plotted in Fig. 8. Again, the result holds not only for $r=0$ but more generally for $r=O(N^{-3/4})$.

6.3. Non-normal Ginibre case

We consider now diffusion initiated from a matrix $(X_0)_{ij}=\alpha\delta_{i,j-1}$. The matrix X_0 has all eigenvalues equal to 0. The initial eigenvalue distribution coincides with the one for the Ginibre case but afterwards the eigenvalue density obeys completely different evolution. Three snapshots of this evolution are shown in Fig. 9. The initial distribution concentrated initially at zero instantaneously expands to a circle of radius $|\alpha|$, which corresponds to the pseudospectrum of the matrix. In the course of evolution the support of the density takes the form of the growing annulus. After a finite time $\tau=|\alpha|^2$ the inner radius of the annulus shrinks to zero and eigenvalues fill up a full disk.

Let us show this by direct calculations. The matrix $\mathcal{M}=(z-X_0)(z-X_0)^\dagger+r^2$ (35) has for our choice of X_0 a tridiagonal form

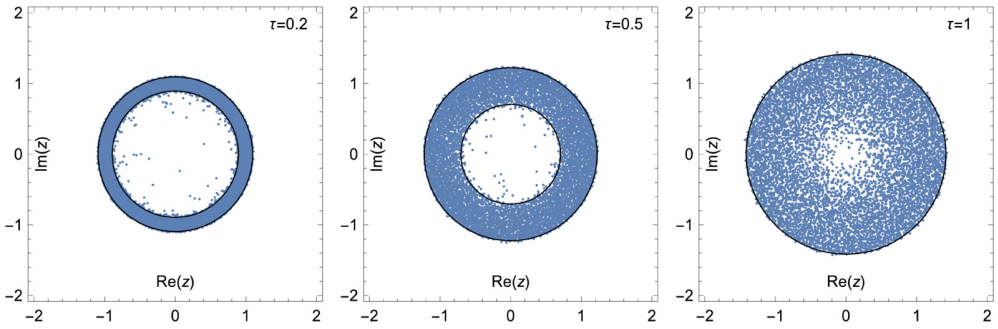

Fig. 9. Evolution of the spectral density for non-normal initial condition $(X_0)_{ij} = \delta_{i,j-1}$, with time snapshots at $\tau = 0.2$, $\tau = 0.5$ and $\tau = 1$, the ensemble consisted of 6 matrices of size $N = 1500$.

$$\mathcal{M} = \begin{pmatrix} a & b & 0 & \dots & 0 \\ \bar{b} & a & b & & 0 \\ \vdots & \bar{b} & \ddots & b & \vdots \\ 0 & & \bar{b} & a & b \\ 0 & \dots & 0 & \bar{b} & d \end{pmatrix} \tag{73}$$

with $a = |z|^2 + r^2 + |\alpha|^2$, $b = -\bar{z}\alpha$ and $d = |z|^2 + r^2$. The determinant of this matrix can be calculated explicitly: $\det \mathcal{M} = \frac{1}{\Delta}\left(d(a_+^N - a_-^N) - |b|^2(a_+^{N-1} - a_-^{N-1}\right)$, where $\Delta = \sqrt{a^2 - 4|b|^2}$ and $a_\pm = \frac{1}{2}(a \pm \Delta)$. The initial effective potential for large N is

$$\phi_0(z, r) = \frac{1}{N}\mathrm{Tr}\log\mathcal{M} \approx \ln a_+ = \ln\frac{1}{2}\left(a + \sqrt{a^2 - 4|z|^2|\alpha|^2}\right) \tag{74}$$

where $a = |z|^2 + r^2 + |\alpha|^2$. We have neglected $1/N$ terms which disappear in the limit $N \to \infty$. The value r_* which maximizes the expression in the Hopf–Lax formula (26) can be calculated from equation (34). For $r = 0$ we get

$$r_* = \begin{cases} 0 & \text{for } z \notin \mathcal{A} \\ \sqrt{T_\alpha - |z|^2 - |\alpha|^2} & \text{for } z \in \mathcal{A} \end{cases}, \tag{75}$$

where we used the notation $T_\alpha = \sqrt{\tau^2 + 4|\alpha|^2|z|^2}$. The boundary of the annulus $\mathcal{A}$ is given by the radii

$$|z|_\pm = \sqrt{|\alpha|^2 \pm \tau}. \tag{76}$$

We see that at the beginning of the evolution, $\tau \approx 0$, the annulus is infinitely narrow forming a one-dimensional circular pseudospectrum. On the other hand for $\tau \to |\alpha|^2$ the inner radius tends to zero and the annulus becomes a disk. Inserting r_* to (26) we obtain the potential

$$\phi(z, r = 0, \tau) = \begin{cases} \ln|\alpha|^2 & \text{for } |z| \in (0; |z|_-) \\ \ln\left(\frac{\tau}{2} + \frac{1}{2}T_\alpha\right) + \frac{|z|^2+|\alpha|^2}{\tau} - \frac{T_\alpha}{\tau} & \text{for } |z| \in (|z|_-; |z|_+) \\ \ln|z|^2 & \text{for } |z| \in (|z|_+; \infty) \end{cases}, \tag{77}$$

for the three regions of the annulus. The spectral density is obtained from the Poisson equation (36) by differentiation the effective potential twice

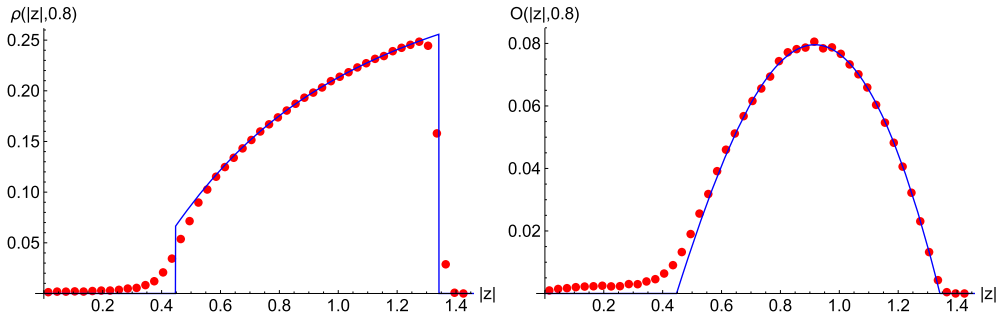

Fig. 10. Numerical analysis of the eigenvector correlator (right plot) and the spectral density (left plot) for the non-normal initial condition, the ensemble consisted of $3 \cdot 10^3$ matrices of size $N = 1000$ with parameters $\alpha = 1$ and time $\tau = 0.8$.

$$\rho(z,\tau) = \frac{1}{\pi\tau}\left(1 - \frac{|\alpha|^2}{T_\alpha}\right) \tag{78}$$

on the support of the annulus and zero otherwise. Using (39) we obtain the eigenvector correlation function to the support of the annulus

$$O(z,\tau) = \frac{1}{\pi\tau^2}\left(T_\alpha - |z|^2 - |\alpha|^2\right). \tag{79}$$

In Fig. 10 we compared the prediction of the two formulas given above with numerical simulations. The agreement is good. Deviations are observed only close to the boundaries and can be attributed to finite-size effects. It is easy to check that all the expressions transform to the corresponding expressions for the Ginibre evolution when $\alpha \to 0$.

Let us shortly discuss finite N effects for the averaged extended characteristic polynomial. We are looking for an universal behavior near the origin for $\tau \to 0$. Around this space–time point an instantaneous transition from N eigenvalues positioned at the origin to the ring of radius $|\alpha|$ happens. In our example the characteristic polynomial is given explicitly as

$$D(z,r,\tau) = \frac{2N}{\tau}\int_0^\infty r' \exp\left(-N\frac{r^2+r'^2}{\tau}\right) I_0\left(\frac{2Nrr'}{\tau}\right) \det\mathcal{M}(z,r')\mathrm{d}r' \tag{80}$$

We consider the following scaling around the origin

$$r' = \theta N^{-7/6}, \qquad |z| = xN^{-1/6}, \tag{81}$$

$$|\alpha| = xN^{-1/6}, \qquad \tau = tN^{-4/3}. \tag{82}$$

Inserting these formulas to the equation above we find an asymptotic function for large N and for $|\alpha| = xN^{-1/6}$:

$$D(z = xN^{-1/6}, r = 0, \tau = tN^{-4/3}) \sim \frac{t}{2} + \frac{t\sqrt{\pi t}}{4x}\exp\left(\frac{t}{4x^2}\right)\operatorname{erfc}\left(-\frac{\sqrt{t}}{2x}\right). \tag{83}$$

We see that it reveals a non-perturbative divergent character near $x = 0$. For the normal Ginibre case we do not find the second term and therefore we identify it as a consequence of the non-normal initial condition. Concluding this section, we note that formula (78) was obtained in [22] for an initial matrix

$$X_0^C = |\alpha| \mathrm{diag}(1, e^{2\pi i/N}, e^{4\pi i/N} \dots e^{2\pi i(N-1)/N}), \tag{84}$$

with N initial equidistant eigenvalues lying on a circle of radius $|\alpha|$. This result is not surprising since the matrix X_0^C is unitarily equivalent to a circulant matrix

$$(X_0^C)_{ij} \sim (X_0)_{ij} + \alpha \delta_{i,N} \delta_{j,1}, \tag{85}$$

which in turn differs from the non-normal initial matrix considered in this paper by just one element whose effect can be neglected in the large N limit. We also mention that finite N spectral densities and large deviations have been recently studied in [33].

7. Spectral stochastic equations

An interesting issue arises, if the stochastic dynamics of the elements of non-hermitian random matrices corresponds to some stochastic equation for the spectra of the underlying matrix. Such a phenomenon holds in the case of Gaussian Unitary Ensembles (GUE), as well as in the case of Circular Unitary Ensembles (CUE), as shown in the seminal paper by Dyson [2]. In the simpler case of GUE, the corresponding Langevin equation for real eigenvalues reads

$$\mathrm{d}\lambda_i = \mathrm{d}B_i + \sum_{i \neq j} \frac{\mathrm{d}t}{\lambda_i - \lambda_j}, \tag{86}$$

where $\mathrm{d}B_i$ reflects the Brownian dynamics and the second, "drift" term comes from the Van der Monde determinant. We neglect the optional Ornstein–Uhlenbeck term $-\lambda_i \, \mathrm{d}t$ on the r.h.s. of (86) since it only contributes to freezing the diffusing end-points of the spectra in the stationary limit. The corresponding Smoluchowski–Fokker–Planck equation written for the resolvent $G(z, \tau)$ of the spectral density $\rho(\lambda, t)$, takes (after rescaling the time $\tau = Nt$ and in the large N limit) the form of complex, inviscid Burgers equation, i.e. eq. (2) [9,34].

Equations for non-hermitian Gaussian ensemble obtained in this work exhibit structural similarity to the hermitian "Burgulence", hence a question arises, if some stochastic dynamics for complex eigenvalues does exist as well. Recently, Osada [17] had examined a stochastic two-dimensional Coulomb system defined as

$$\mathrm{d}\lambda_i = \mathrm{d}B_i^{(2)} + \sum_{i \neq j} \frac{\mathrm{d}t}{\bar{\lambda}_i - \bar{\lambda}_j} = \mathrm{d}B_i^{(2)} + \sum_{i \neq j} \frac{\lambda_i - \lambda_j}{|\lambda_i - \lambda_j|^2} \mathrm{d}t, \tag{87}$$

where $\mathrm{d}B_i^{(2)}$ is a two-dimensional (complex) Brownian walk. In particular, he has shown that such interacting Brownian motion equipped with trivial initial condition (all λ_i put to zero at $t = 0$) leads to limiting distribution of λ_i representing a uniform disk, therefore resembling the Ginibre Ensemble spectrum.

From the point of view of our analysis this result is curious, since we have proven that the dynamics of eigenvalues is intimately connected to the dynamics of eigenvectors, which seem to decouple completely from "hypothetical" eigenvalues of Ginibre ensemble in (87). We suggest the resolution of this puzzle.

It is well known, that in so-called Normal Random Matrix model [35] with axially symmetric potentials all correlation functions can be expressed in terms of holomorphic functions of a single variable. Moreover, the exact integrability of such models can be linked to $(2 + 1)$-dimensional Burgers equations. Quite remarkably, in the case of the potential $V(z, \bar{z}) = |z|^2$, the correlations

of the Normal Random Matrix model are identical to the correlations of the Ginibre Ensemble [36]. We therefore conjecture, that the stochastic equation (87) corresponds rather to a Gaussian Normal Random Matrix model than to the Ginibre Ensemble.

Because normal matrices are diagonalizable by a single unitary transformation, alike the hermitian matrices, the eigenvectors decouple from the eigenvalues which explains the lack of these degrees of freedom in (87) and its formal similarity to hermitian stochastic equation (86). Based on this and the significance of eigenvectors presented in this paper we find it highly probable that the Ginibre-like dynamics of (87) is accidental and proper stochastic equation governing the dynamics of non-hermitian random matrices is not known.

The speculative links between our approach for generic complex matrices and non-Gaussian Normal Random Matrix models represent a challenge, which we plan to address in the forthcoming publications.

8. Conclusions

We have shown that a consistent description of non-hermitian Gaussian ensemble requires the knowledge of the detailed dynamics of co-evolving eigenvalues and eigenvectors. Moreover, the dynamics of eigenvectors seems to play a superior role (at least in the $N \to \infty$ limit) and leads directly to the inference of the spectral properties. This is a dramatically different scenario as compared to the standard random matrix models, where the statistical properties of eigenvalues are of primary importance, and the properties of eigenvectors are basically trivial due to the their decoupling from the spectra. We have considered examples of Ginibre, spiric section and non-normal Ginibre where the formulas for spectral density, the 1-point eigenvector correlation function, electrostatic potential and universal functions were obtained and positively crosschecked with numerical simulations. By studying the dynamics of characteristics we anticipated the novel universality obtained for the spiric case as an error function type. We obtained compact formulas for both spectral density and eigenvector correlator for which the latter unraveled a promising determinantal structure. The diffusion equation, as an equation exact for finite N, was used mainly to obtain the universal behavior.

We conjecture that the hidden dynamics of eigenvectors discovered in this work and described for the Gausssian non-hermitian ensemble, is a general feature of all non-hermitian random matrix models, and has to appear systematically in $1/N$ expansion.

Our formalism could be exploited to expand the area of application of non-Hermitian random matrix ensembles within problems of growth, charged droplets in quantum Hall effect and gauge theory/geometry relations in string theory beyond the subclass of complex matrices represented by normal matrices.

One of the challenges is an explanation, why, despite being so different, the Smoluchowski–Fokker–Planck equations for hermitian and non-hermitian random matrix models exhibit structural similarity to simple models of turbulence, where so-called Burgers equation plays the vital role, establishing the flow of the spectral density of eigenvalues in the case of the hermitian or unitary ensembles and the flow of certain eigenvector correlator in the case of non-hermitian ensembles.

We believe that our findings will contribute to understanding of several puzzles of non-hermitian dynamics, as for instance extreme sensitivity of spectra of non-hermitian systems to perturbations [24,27]. We also hope that the quaternion extension used in our paper may help to understand better the mathematical subtleties of the measure of non-hermitian operators [37].

Acknowledgements

M.A.N., J.G., Z.B., W.T. and P.W. are supported by the Grant DEC-2011/02/A/ST1/00119 of the National Centre of Science. M.A.N. thanks Neil O'Connell for pointing out Ref. [17] and appreciates discussions with Neil O'Connell, Boris Khoruzhenko, Andu Nica, Dima Savin, Roger Tribe and Oleg Zaboronski. The authors are also grateful to Charles Bordenave for pointing the relevant Ref. [33] on mathematical subtleties of normal Gaussian models. J.G. thanks prof. Thomas Guhr for the warm hospitality at the University Duisburg–Essen where part of this work was done.

Appendix A. Derivation of the diffusion equation (20)

In this appendix we demonstrate that the averaged extended characteristic polynomial (19) obeys the diffusion equation (20). The determinant in (19) can be expressed using Grassmann variables as

$$D(z,w,\tau) = \int \mathcal{D}[X]\mathcal{D}[\eta,\xi]P(X,\tau)\exp T_G(X,z,w;\eta,\xi), \tag{A.1}$$

with the object T_G given by

$$\begin{aligned} T_G(X,z,w;\eta,\xi) = &\sum_{i,j}\left(-x_{ij}(\bar\eta_i\eta_j+\bar\xi_j\xi_i) - iy_{ij}\left(\bar\eta_i\eta_j-\bar\xi_j\xi_i\right)\right) \\ &+\sum_i\left(z\bar\eta_i\eta_i + \bar z\bar\xi_i\xi_i + w\bar\xi_i\eta_i - \bar w\bar\eta_i\xi_i\right), \end{aligned} \tag{A.2}$$

where $\eta, \bar\eta, \xi$ and $\bar\xi$ are Grassmann variables and $X_{ij} = x_{ij} + iy_{ij}$. With the help of heat equation (5) for the joint probability density function $P(X,\tau)$ we obtain

$$\begin{aligned} \partial_\tau D(z,w,\tau) &= \int \mathcal{D}[X,\eta,\xi]\,(\partial_\tau P)\exp T_G \\ &= \frac{1}{4N}\int \mathcal{D}[X,\eta,\xi]\left(\sum_{i,j}\left(\partial^2_{x_{ij}}+\partial^2_{y_{ij}}\right)P\right)\exp T_G \\ &= \frac{1}{4N}\int \mathcal{D}[X,\eta,\xi]P\left(\sum_{i,j}\left(\partial^2_{x_{ij}}+\partial^2_{y_{ij}}\right)\exp T_G\right) \\ &= \frac{1}{N}\int \mathcal{D}[X,\eta,\xi]P\sum_{ij}\bar\eta_i\eta_j\bar\xi_j\xi_i\exp T_G, \end{aligned} \tag{A.3}$$

where we integrated by parts twice and we used

$$\partial^2_{x_{ij}}\exp T_G = (\bar\eta_i\eta_j+\bar\xi_j\xi_i)(\bar\eta_i\eta_j+\bar\xi_j\xi_i)\exp T_G = 2\bar\eta_i\eta_j\bar\xi_j\xi_i\exp T_G, \tag{A.4}$$

$$\partial^2_{y_{ij}}\exp T_G = (\bar\eta_i\eta_j+\bar\xi_j\xi_i)(\bar\eta_i\eta_j+\bar\xi_j\xi_i)\exp T_G = 2\bar\eta_i\eta_j\bar\xi_j\xi_i\exp T_G. \tag{A.5}$$

On the other hand we have

$$\partial_{w\bar w}\exp T_G = \sum_{i,j}\bar\eta_i\eta_j\bar\xi_j\xi_i\exp T_G, \tag{A.6}$$

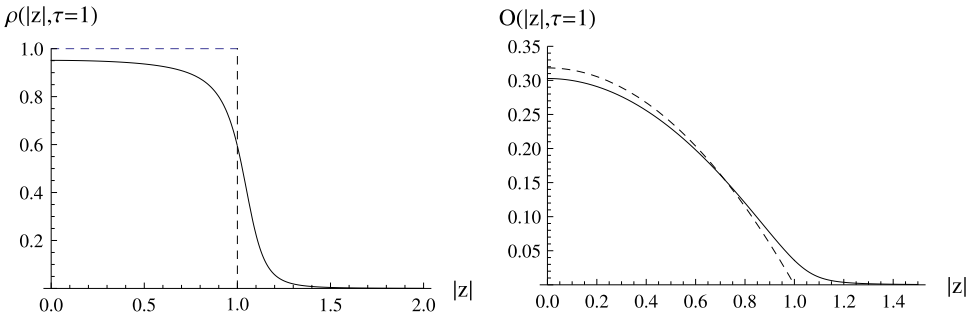

Fig. 11. The large N spectral density ρ (left plot) and eigenvector correlator O (right plot) for $r = 0.05$ (solid line) and $r = 0$ (dashed line).

from which it follows that

$$\frac{1}{N}\partial_{w\bar{w}} D(z,w,\tau) = \frac{1}{N}\int \mathcal{D}[X,\eta,\xi] P \sum_{ij} \bar{\eta}_i \eta_j \bar{\xi}_j \xi_i \exp T_G. \tag{A.7}$$

We see that the expressions on the right hand side of (A.3) and (A.7) are identical and thus we have $\frac{1}{N}\partial_{w\bar{w}} D(z,w,\tau) = \partial_\tau D(z,w,\tau)$ (20). As a side remark we note that this calculation can be almost verbatim repeated for the averaged extended "inverse" characteristic polynomial

$$F(z,w,\tau) = \left\langle \det \begin{pmatrix} z - X & -\bar{w} \\ w & \bar{z} - X^\dagger \end{pmatrix}^{-1} \right\rangle_\tau, \tag{A.8}$$

which obeys the diffusion equation in the opposite time direction $\tau \to -\tau$

$$-\partial_\tau F = \frac{1}{N}\partial_{w\bar{w}} F. \tag{A.9}$$

Appendix B. The $r \neq 0$ regime as a Wishart/chiral deformation

The majority of results discussed in this paper are confined to the "physical" region where $r \to 0$. One can however keep the parameter r nonzero. In the Coulomb gas interpretation, this deformation introduces a complex nonlinear interaction between the eigenvalues of unknown interpretation.

We consider the Ginibre case. The expression in the Hopf–Lax equation (26)

$$\phi(z,r,\tau) = \max_{r'} \left(\ln(|z|^2 + r'^2) - \frac{(r'-r)^2}{\tau} \right), \tag{B.1}$$

is maximized by r_* obeying a cubic equation

$$(r_* - r)(r_*^2 + |z|^2) = r_* \tau. \tag{B.2}$$

The corresponding spectral density $\rho(z,r,\tau)$ and the eigenvector gradient $v(z,r,\tau)$ are shown in Fig. 11. There are no critical points for $r \neq 0$ and instead we see a smooth crossover between two phases. This is in accordance with the fact that the shock is present only on the $r = 0$ plane. Consider the formula for the averaged extended characteristic polynomial for the Ginibre evolution

$$D(z,r,\tau)=\frac{2N}{\tau}\int_0^\infty r'\exp\left(-N\frac{r^2+r'^2}{\tau}\right)I_0\left(\frac{2Nrr'}{\tau}\right)\left(|z|^2+r'^2\right)^N\mathrm{d}r'. \tag{B.3}$$

Applying the Newton binomial formula to $\left(|z|^2+r'^2\right)^N$ a using the following integral representation of Laguerre polynomials

$$\int_0^\infty dqq^{2k+1}e^{-\frac{N}{\tau}q^2}I_0\left(\frac{2Nrq}{\tau}\right)=\frac{\tau}{2N}\left(\frac{\tau}{N}\right)^k k!L_k\left(-\frac{N}{\tau}r^2\right)e^{\frac{N}{\tau}r^2}, \tag{B.4}$$

we get

$$D(z,r,\tau)=e^{-\frac{N}{\tau}r^2}N!\left(\frac{\tau}{N}\right)^N\sum_{k=0}^N\frac{1}{k!}L_{N-k}\left(-\frac{Nr^2}{\tau}\right)\left(\frac{N|z|^2}{\tau}\right)^k. \tag{B.5}$$

It is not surprising that Laguerre polynomials show up in this context since the Ginibre ensemble is closely related to the Wishart/chiral ensemble where they naturally occur. In fact by recalling the definition (19)

$$D(z,r,\tau)=\left\langle r^2+(z-X)(\bar{z}-X^\dagger)\right\rangle_\tau \tag{B.6}$$

we see that for $z\to 0$ the object within the brackets is a Wishart matrix and

$$D(z=0,r,\tau)=e^{-\frac{N}{\tau}r^2}N!\left(\frac{\tau}{N}\right)^N L_N\left(-\frac{N}{\tau}r^2\right). \tag{B.7}$$

For $r\to 0$ we have $D(z=0,r=0,\tau)=N!\left(\frac{\tau}{N}\right)^N$.

Appendix C. The kernel structure

We argue that in the case of Ginibre matrices, the characteristic polynomial D in the $r\to 0$ limit has essentially the same information as the microscopic kernel of the underlying determinantal process. This stems from the observation made in [39] where the n-point correlation function of a Ginibre matrix model is related to a random matrix QCD partition function with n quarks with appropriately decreased matrix size:

$$\det(K_N(z_i,z_j))_{i,j=1\ldots n}=c(z_1,\ldots,z_n)\left\langle\prod_{i=1}^n\det(z_i-X)\det(\bar{z}_i-X^\dagger)\right\rangle_{X_{N-n}}, \tag{C.1}$$

where $\det(K_N(z_i,z_j))_{i,j=1\ldots n}=\left\langle\prod_{i=1}^n\mathrm{Tr}\delta^2(z_i-X)\right\rangle_{X_N}$ is the correlation function averaged over N dimensional matrix and c denotes a known z dependent proportionality factor.

We study closer the $n=1$ case because for this parameter the prefactor c is the weight function $w(z)$ and the averaged determinants are exactly the characteristic polynomial D in the $r\to 0$ limit. Therefore, in this particular case, we obtain an on-diagonal kernel formula

$$K_N(z,z)=C_N w(z)D^{(N-1)}(z,r=0,\tau), \tag{C.2}$$

with some numerical constant C_N. The off-diagonal kernel is not so easily obtainable by the above formula however we make an educated guess based on the symmetry of arguments and the result of Akemann and Vernizzi [38]. We write the full kernel as

$$K_N(z,v) = C_N\sqrt{w(z)}\sqrt{w(v)}D^{(N-1)}([z,v], r=0,\tau), \tag{C.3}$$

where the two-argument characteristic polynomial D was created by substituting $|z|^2 \to z\bar{v}$:

$$D^{(N-1)}([z,v],r,\tau) = \frac{2N}{\tau}\int_0^\infty q\exp\left(-N\frac{q^2+r^2}{\tau}\right)I_0\left(\frac{2Nqr}{\tau}\right)(q^2+z\bar{v})^{N-1}\mathrm{d}q, \tag{C.4}$$

which for $r=0$ it is readily solved as

$$D^{(N-1)}([z,v],r=0,\tau) = \left(\frac{\tau}{N}\right)^{N-1}\Gamma(N)\sum_{k=0}^{N-1}\left(\frac{Nz\bar{v}}{\tau}\right)^k\frac{1}{k!}. \tag{C.5}$$

To complete the argument demonstrating that the full information resides in the characteristic polynomial D we will find an a priori unknown weight function $w(z)$ present in formula (C.3) from the D alone. This is done by using decomposition in the biorthogonal basis

$$D^{(N-1)} = \sum_{k,l=0}^{N-1} c_{kl}X_k(z)\bar{Y}_l(v), \tag{C.6}$$

where X_i, Y_i are biorthogonal polynomials with respect to an unknown measure W

$$\int d^2z W(z)X_i(z)\bar{Y}_j(z) = g_{ij}. \tag{C.7}$$

The bi-orthogonality matrix g_{ij} is the inverse of the matrix built of expansion coefficients c_{ij}. We proceed by finding a basis X_i, Y_j in which c_{kl} are diagonal. Then we infer the formula for W by considering its moments (C.7). We start with a formula (C.5) which is already in a diagonal form with

$$X_k(z) = z^k, \qquad Y_k(v) = v^k, \qquad c_{kk} = \frac{N!}{k!}\left(\frac{\tau}{N}\right)^{N-1-k}. \tag{C.8}$$

The moments are therefore given by

$$\int d^2z\, W(z)|z|^{2k} = \frac{k!}{N!}\left(\frac{N}{\tau}\right)^{N-1-k}. \tag{C.9}$$

By assuming the radial symmetry and setting $|z|^2 = p$ we obtain

$$\int_0^\infty dpW(\sqrt{p})p^k = \frac{(N/\tau)^N}{\pi N!}\frac{k!}{(N/\tau)^{k+1}}, \tag{C.10}$$

from which the characteristic function is given by

$$M_W(t) = \int_0^\infty dpW(\sqrt{p})e^{itp} = \frac{(N/\tau)^{N-1}}{\pi N!}\left(1-\frac{it\tau}{N}\right)^{-1}. \tag{C.11}$$

It is exactly the characteristic function of an exponential distribution $\lambda e^{-\lambda x}$ with $\lambda = N/\tau$

$$W(\sqrt{p}) = \frac{(N/\tau)^N}{\pi N!}e^{-\frac{N}{\tau}p}. \tag{C.12}$$

This procedure gives the unknown weight $w(z) = e^{-N/\tau|z|^2}$ along with the normalization coefficient $C_N = \frac{1}{\pi N!}\left(\frac{N}{\tau}\right)^N$

$$W(|z|^2) = C_N w(z). \tag{C.13}$$

The kernel is therefore equal to

$$K_N(z,v) = \frac{1}{\tau\pi}\exp\left(-\frac{N}{2\tau}(|z|^2+|v|^2)\right)\sum_{k=0}^{N-1}\left(\frac{Nz\bar{v}}{\tau}\right)^k\frac{1}{k!}. \tag{C.14}$$

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DIFFUSION IN THE SPACE OF COMPLEX HERMITIAN MATRICES — MICROSCOPIC PROPERTIES OF THE AVERAGED CHARACTERISTIC POLYNOMIAL AND THE AVERAGED INVERSE CHARACTERISTIC POLYNOMIAL*

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(Received January 31, 2015)

We show that the averaged characteristic polynomial and the averaged inverse characteristic polynomial, associated with the Hermitian matrices whose elements perform a random walk in the space of complex numbers, satisfy certain partial differential, diffusion-like equations. These equations are valid for matrices of arbitrary size and for any initial condition assigned to the process. The solutions have compact integral representation that allows for a simple study of their asymptotic behavior, uncovering the Airy and Pearcey functions.

DOI:10.5506/APhysPolB.46.1801
PACS numbers: 05.40.–a, 47.27.tb, 02.10.Yn

1. Introduction

As two of us have argued [1], a particularly interesting, hydrodynamic-like picture of the spectral evolution emerges if one exploits Dyson's idea [2] of introducing temporal dynamics into random matrix ensembles. Such dynamics appears in random matrix models in a broad context: in the physical

* Presented at the Conference "Random Matrix Theory: Foundations and Applications", Kraków, Poland, July 1–6, 2014.

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applications, "time" can correspond to the length of a mesoscopic wire, area of the string/Wilson loop or the inverse temperature (see *e.g.* [3]): in mathematical studies, such dynamics appears in the study in non-intersecting Brownian paths and in relations to random skew planar partitions [4–6], to mention a few.

Here, we consider perhaps the simplest case of random matrix model, *i.e.* a Hermitian, $N \times N$ matrix whose entries perform a properly normalized, continuous random walk in the space of complex numbers. Let this process be initiated with a matrix filled with zeros. For this particular choice, a time-dependent, monic Hermite polynomial satisfies a complex diffusion equation with a diffusion constant equal to $-\frac{1}{2N}$. A key feature of this setting is that the polynomial is equal to the averaged characteristic polynomial (hereafter ACP) associated with the random matrix. The latter can be, moreover, transformed into a different function (by taking its logarithmic derivative) which, in turn, fulfills the viscid Burgers equation. In the large N (inviscid) limit, it admits solutions exhibiting shocks, whose positions coincide with the edges of the eigenvalue spectrum. In the finite N (viscid) case, one can perform an expansion around the shocks to obtain the well-known Airy asymptotic behavior of the orthogonal polynomial.

This analysis can be carried over to the case of diffusing Wishart matrices [7], for which the relevant monic orthogonal polynomials are given in terms of the Laguerre polynomials, corresponding to trivial initial condition for the evolution. We have shown [8] that for the evolution corresponding to non-trivial initial conditions, one can also rely directly on the ACP, despite it is not equal in this case to the orthogonal polynomial. By representing the determinant as a Berezin integral, we have derived the complex diffusion equation independently of the initial conditions, which allowed us to study a new, microscopic universal behavior at the spectral shock reaching the so-called wall at the origin (resulting in a Bessoid type function).

In this paper, we demonstrate that this strategy works also for the simplest case of Hermitian matrices, *i.e.* Gaussian Unitary Ensemble (hereafter GUE). In this way, we fill the certain logical gap between our early paper on GUE [1] and recent paper of Wishart ensemble [8]. In particular, we demonstrate that both the ACP and the averaged inverse characteristic polynomial (AICP) for GUE satisfy the same complex diffusion equation, except that for the former the diffusion constant is equal to $-\frac{1}{2N}$, whereas for the latter it is $\frac{1}{2N}$. As in the case of the Wishart ensemble, the new proof works regardless of the actual form of the initial condition imposed on the process and thus allows us to examine two different, generic scenarios. The solution of the diffusion equation leads to simple integral representations for both the ACP and the AICP, which makes it possible to study their asymptotic, large N, universal behaviors. In particular, we recover the known scaling

property of the ACP and the AICP at the edge of the spectrum in terms of Airy functions. Moreover, for a process initiated with at least two distinct eigenvalues, a case we were not able to study within the orthogonal polynomials based approach, when two edges of the spectrum meet, the Pearcey functions emerge. Thus, the diffusion scenario provides a natural and simple way to re-derive the universal functions corresponding to the fold and cusp singularities in random matrix models. Certainly, some of the results presented in this paper are not new and have been derived by other methods by several authors. One can consider, therefore, this work as a pedagogical review of finite and infinite N effects in GUE, perhaps from a non-orthodox point of view. However, we believe that the scheme we are proposing here for GUE has much greater potential for broad class of random matrix models. Indeed, some recent results mentioned in the conclusions have already confirmed this rationale.

The paper is organized as follows. We start by introducing the stochastic evolution of the studied matrix. Using the representations of the determinant and its inverse as Gaussian integrals over, respectively, Grassmann or complex variables, we derive the diffusion equations for the ACP and the AICP. For the simplest scenario, in which the process is initialized with a null matrix, we crosscheck the equation for the AICP by exhibiting the equivalence of its solution with the Cauchy transform of the ACP. In such a way, we establish a connection to the well-known result of time-dilated Hermite polynomial and its Cauchy transform. In the following section, we derive the corresponding Burgers equation, which we solve in the large N limit with the method of complex characteristics and obtain the associated Green's function for two different generic examples. In the first one, the initial matrix is filled with zeros, whereas in the second, it has two distinct non-vanishing eigenvalues. We subsequently use the saddle point method to inspect how the ACP and AICP behave in the former scenario, at the points corresponding to the edges of the probability density function for the eigenvalues, asymptotically when N is increased. Whereas in the case of trivial initial conditions our result is equivalent to the well-known case of Airy asymptotics for Hermite polynomials, the case of non-trivial initial conditions offers a new perspective for the case of the so-called Pearcey kernel. Since the spectrum forms then two disjoint lumps of eigenvalues that eventually collide, more subtle saddle point analysis around the time and point of this collision is required. In particular, one has to consider rather refine coalescence of three saddle points. The studies of Pearcey asymptotics in the literature are either based on the introduction of the biorthogonal ensemble [22], or by application of the powerful Riemann–Hilbert approach [4]. We found it quite amusing, that a simple diffusion equation (or equivalent to it viscid Burgers equation) can lead in a straightforward way to both cases of universality. Moreover, all the subtleties of the double scaling limit in the

vicinity of the critical points (here shock waves) are put in a nut-shell by the emergence of a spectral viscosity $1/2N$ in the complex Burgers equation. Finally, we make a link to the well-studied case of random matrix kernel for GUE, showing how the ACP and AICP can be used to reconstruct the kernel. The last section summarizes our results and puts them in a broader context.

2. Diffusion of Hermitian matrices

Let us introduce an $N \times N$ Hermitian matrix H by defining its complex entries according to

$$H_{ij} = \begin{cases} x_{ii}\,, & i = j\,, \\ \frac{1}{\sqrt{2}}(x_{ij} + iy_{ij})\,, & i \neq j\,, \end{cases} \tag{1}$$

where $x_{ij} = x_{ji}$ and $y_{ij} = -y_{ji}$, with x_{ij} and y_{ij} real. Furthermore, let x_{ij} and y_{ij} perform white-noise driven, independent random walks, such that

$$\langle \delta H_{ij} \rangle = 0\,, \qquad \left\langle (\delta H_{ij})^2 \right\rangle = \frac{1}{N}\delta\tau \tag{2}$$

for any i and j. Let $P(x_{ij},\tau)P(y_{ij},\tau)$ be the probability that the off-diagonal matrix entry H_{ij} will change from its initial state to $\frac{1}{\sqrt{2}}(x_{ij} + iy_{ij})$ after time τ. Analogically, $P(x_{ii},\tau)$ is the probability of the diagonal entry H_{ii} becoming equal to x_{ii} at τ. The evolution of these functions is governed by the following diffusion equations:

$$\begin{aligned} \frac{\partial}{\partial\tau}P(x_{ij},\tau) &= \frac{1}{2N}\frac{\partial^2}{\partial x_{ij}^2}P(x_{ij},\tau)\,, \\ \frac{\partial}{\partial\tau}P(y_{ij},\tau) &= \frac{1}{2N}\frac{\partial^2}{\partial y_{ij}^2}P(y_{ij},\tau)\,, \qquad i \neq j\,. \end{aligned} \tag{3}$$

Moreover, the joint probability density function

$$P(x,y,\tau) \equiv \prod_k P(x_{kk},\tau)\prod_{i<j} P(x_{ij},\tau)P(y_{ij},\tau) \tag{4}$$

satisfies the following equation [9]

$$\begin{aligned} \partial_\tau P(x,y,\tau) &= \mathcal{A}(x,y)P(x,y,\tau)\,, \\ \mathcal{A}(x,y) &= \frac{1}{2N}\sum_k \frac{\partial^2}{\partial x_{kk}^2} + \frac{1}{2N}\sum_{i<j}\left(\frac{\partial^2}{\partial x_{ij}^2} + \frac{\partial^2}{\partial y_{ij}^2}\right). \end{aligned} \tag{5}$$

With the setting thus defined, let us proceed to the derivation of the partial differential equations governing the ACP and AICP.

2.1. Evolution of the averaged characteristic polynomial

Let $U_N(z,\tau)$ be the averaged characteristic polynomial associated with the diffusing matrix H: $U_N(z,\tau) \equiv \langle \det(z-H) \rangle$, where the angular brackets denote the averaging over the time-dependent probability density (4). In order to derive the partial differential equation governing the ACP, we write the determinant as a Gaussian integral over Grassmann variables $\eta_i, \bar{\eta}_i$

$$\det A = \int \prod_{i,j} d\eta_i d\bar{\eta}_j \exp\left(\bar{\eta}_i A_{ij} \eta_j\right). \tag{6}$$

This allows us to express the averaged characteristic polynomial in the following way

$$U_N(z,t) = \int \mathcal{D}[\bar{\eta},\eta,x,y] P(x,y,\tau) \exp\left[\bar{\eta}_i \left(z\delta_{ij} - H_{ij}\right) \eta_j\right], \tag{7}$$

where the joint integration measure is defined by

$$\mathcal{D}[\bar{\eta},\eta,x,y] \equiv \prod_{i,j} d\eta_i d\bar{\eta}_j \prod_k dx_{kk} \prod_{n<m} dx_{nm} dy_{nm}. \tag{8}$$

The Hermiticity condition ($H_{ij} = \bar{H}_{ji}$) allows us to write the argument of the exponent of (7) in a convenient form

$$\begin{aligned} T_f(\bar{\eta},\eta,x,y,z) \;\equiv\; & \sum_r \bar{\eta}_r \left(z - x_{rr}\right) \eta_r \\ & - \frac{1}{\sqrt{2}} \sum_{n<m} \left[x_{nm} \left(\bar{\eta}_n \eta_m - \eta_n \bar{\eta}_m\right) + i y_{nm} \left(\bar{\eta}_n \eta_m + \eta_n \bar{\eta}_m\right)\right]. \end{aligned}$$

Note that the time dependence of $\pi(z,\tau)$ resides entirely in $P(x,y,\tau)$. By differentiating Eq. (7) with respect to τ, and using Eq. (5), one ends up with an expression where the operator $\mathcal{A}(x,y)$ acts on the joint probability density function. Integrating by parts with respect to x_{ij} and y_{ij}, one obtains

$$\partial_\tau U_N(z,\tau) = \int \mathcal{D}[\bar{\eta},\eta,x,y] P(x,y,\tau) \mathcal{A}(x,y) \exp\left[T_f(\bar{\eta},\eta,x,y,z)\right]. \tag{9}$$

At this point, we differentiate with respect to the matrix elements (acting with $\mathcal{A}(x,y)$), exploit some simple properties of the Grassmann variables, and obtain

$$\partial_\tau U_N(z,\tau) = -\frac{1}{N} \int \mathcal{D}[\bar{\eta},\eta,x,y] P(x,y,\tau) \sum_{i<j} \bar{\eta}_i \eta_i \bar{\eta}_j \eta_j \exp\left[T_f(\bar{\eta},\eta,x,y,z)\right]. \tag{10}$$

It is easily verified that this expression, when multiplied by $-2N$, matches the double differentiation with respect to z of Eq. (7). We thus end up with

$$\partial_\tau U_N(z,t) = -\frac{1}{2N}\partial_{zz} U_N(z,\tau)\,. \tag{11}$$

This is the sought for diffusion equation for the ACP. Note that the same equation was already obtained in Ref. [1], albeit for a very specific initial condition, for which $U_N(z,t)$ is a scaled Hermite polynomial. The present derivation has the advantage of being independent of the choice of initial condition.

2.2. Evolution of the averaged inverse characteristic polynomial

We now turn to the averaged inverse characteristic polynomial

$$E_N(z,t) \equiv \left\langle \frac{1}{\det(z-H)} \right\rangle \tag{12}$$

to which we are going to apply a similar strategy. In this case, we use the fact that the inverse of a determinant has a well-known representation in terms of a Gaussian integral over complex variables ξ_i

$$\frac{1}{\det A} = \int \prod_{i,j} d\xi_i d\bar{\xi}_j \exp\left(-\bar{\xi}_i A_{ij} \xi_j\right). \tag{13}$$

As in the ACP case, we use this representation to express (12) as

$$E_N(z,\tau) = \int \mathcal{D}\left[\bar{\xi},\xi,x,y\right] P(x,y,\tau) \exp\left[\bar{\xi}_i \left(H_{ij} - z\delta_{ij}\right)\xi_j\right], \tag{14}$$

where, again, the proper notation for the joint integration measure was introduced. Performing the differentiation with respect to τ yields

$$\partial_\tau E_N(z,\tau) = \int \mathcal{D}\left[\bar{\xi},\xi,x,y\right] P(x,y,\tau)\mathcal{A}(x,y) \exp\left[T_b\left(\bar{\xi},\xi,x,y\right)\right], \tag{15}$$

with

$$\begin{aligned} T_b(\bar{\xi},\xi,x,y,z) \;\equiv\; & \sum_r \bar{\xi}_r \left(x_{rr} - z\right)\xi_r \\ & + \frac{1}{\sqrt{2}} \sum_{n<m} \left[x_{nm}\left(\bar{\xi}_n\xi_m + \xi_n\bar{\xi}_m\right) + i y_{nm}\left(\bar{\xi}_n\xi_m - \xi_n\bar{\xi}_m\right)\right], \end{aligned}$$

where we have used (5), the Hermiticity of H and we have performed integrations by parts. After differentiation with respect to the matrix elements, one obtains

$$\partial_\tau E_N(z,\tau) = \frac{1}{N}\int \mathcal{D}\left[\bar{\xi},\xi,x,y\right] P(x,y,\tau) \times\left(\sum_{i<j}\bar{\xi}_i\xi_i\bar{\xi}_j\xi_j+\frac{1}{2}\sum_k \bar{\xi}_k\xi_k\bar{\xi}_k\xi_k\right)\exp\left[T_b\left(\bar{\xi},\xi,x,y,z\right)\right], \quad (16)$$

which, multiplied by $2N$, matches the double differentiation of Eq. (14) with respect to z. The final result reads

$$\partial_\tau E_N(z,t) = \frac{1}{2N}\partial_{zz}E_N(z,t)\,, \quad (17)$$

the announced diffusion equation for the AICP. This equation is identical to that satisfied by the ACP except for the sign of the diffusion constant.

3. The integral representation

The main advantage of the equations derived above is that they have obvious solutions in terms of initial condition-dependent integrals. In this section, we explicitly state those representations and show additionally how, for the simplest initial condition, one is a Cauchy transform of the other. Let us also note here that these types of integrals were obtained [10] as representations of multiple orthogonal polynomials [11, 12] and, equivalently, as averaged characteristic polynomials of GUE matrices perturbed by a source [13]. We note that the presented integral representations can be also considered as a special case of the more general rations of characteristic polynomials, where supersymmetric methods have to be used [9].

3.1. The averaged characteristic polynomial

One can verify by a direct calculation that the expression

$$U_N(z,\tau) = \mathcal{C}\,\tau^{-1/2}\int_{-\infty}^{\infty}\exp\left(-N\frac{(q-iz)^2}{2\tau}\right)U_N(-iq,\tau=0)\,dq \quad (18)$$

satisfies the complex diffusion equation (11) governing the evolution of the averaged characteristic polynomial. The imaginary unit in the exponent and in the argument of the initial condition arises from the negative value of the diffusion constant in this equation. For finite N, the most general form of

the initial condition is $U_N(z,\tau=0)=\prod_i(z-\lambda_i)$, where the λ_is are real due to the Hermiticity of the initial matrix $H(\tau=0)$. Exploiting the steepest descent method to match this with Eq. (18), one determines the constant term $\mathcal{C}$. The saddle point associated with $\tau\to 0$ is $u_0=iz$. Performing the Gaussian integration around the saddle point, we obtain $\mathcal{C}=\sqrt{\frac{N}{2\pi}}$ so that Eq. (18) reads

$$U_N(z,\tau)=\sqrt{\frac{N}{2\pi\tau}}\int_{-\infty}^{\infty}\exp\left(-N\frac{(q-iz)^2}{2\tau}\right)U_N(-iq,\tau=0)\,dq\,. \tag{19}$$

3.2. The averaged inverse characteristic polynomial

The integral representation of the averaged inverse characteristic polynomial arising as a solution to the partial differential equation (17) is

$$E_N(z,\tau)=\mathcal{C}\int_{\Gamma}\exp\left(-N\frac{(q-z)^2}{2\tau}\right)E_N(q,\tau=0)\,dq\,. \tag{20}$$

As in the case of the ACP, the initial condition has to be recovered. Here however, $E_N(z,\tau=0)$ has poles on the real axis and the contour Γ must avoid these poles. A first possibility is to choose Γ_+ parallel and slightly above the real axis. In this case, the saddle point analysis for $\tau\to 0$ is performed by moving Γ_+ upward so that it crosses the saddle point $q_0=z$. Obviously, this is possible only if $\mathrm{Im}z>0$. If instead $\mathrm{Im}z<0$, we need to choose Γ_- also parallel to the real axis but slightly below. Imposing an integration contour that would switch from the upper to the lower half plane (and *vice versa*) in between the poles, would results in a function no longer being the solution of the initial problem.

In the simple case of $U_N(z,\tau=0)=z^N$, we can cross check the above results using the well-known [14] Cauchy transform formula linking the ACP and the AICP. In particular, (19) coincides with the integral representation of the Hermite polynomial [15]

$$\pi_k(s,\tau)=(-i)^k\sqrt{\frac{N}{2\pi\tau}}\int_{-\infty}^{\infty}q^k\exp\left(-\frac{N}{2\tau}(q-is)^2\right)dq \tag{21}$$

and the aforementioned Cauchy transform formula reads

$$E_N(z,\tau)=\frac{1}{c_{N-1}^2}\int\frac{ds}{z-s}\pi_{N-1}(s,\tau)\exp\left(-\frac{Ns^2}{2\tau}\right)\,, \tag{22}$$

where the constant $c_k^2 = \sqrt{\frac{2\pi\tau}{N}}\left(\frac{\tau}{N}\right)^k k!$ is the normalization of the monic polynomials. Note that this is valid only for this particular, simplest initial condition. Analogical prescriptions for other cases are significantly more complicated [11].

To proceed, we plug (21) into (22). After transforming the q^{N-1} term into a differentiation of the exponent with respect to s, followed by integrating by parts, we get the result

$$E_N(z,\tau) = \sqrt{\frac{N}{2\pi\tau}} \int\limits_{-\infty+z}^{\infty+z} \frac{1}{u^N} \exp\left(-N\frac{(u-z)^2}{2\tau}\right) du\,. \tag{23}$$

The integration contour can be, in turn, deformed to the real axis and a half circle enclosing the pole at 0 from above ($\mathrm{Im} z > 0$) or below ($\mathrm{Im} z < 0$) — in complete agreement with equation (20).

4. Large N spectral dynamics

Let λ_is be the eigenvalues of the diffusing matrix $H(\tau)$. The connection between the spectral density $\rho(\lambda) \equiv \langle \sum_i \delta(\lambda - \lambda_i)\rangle$, in the limit of N going to infinity and the averaged characteristic polynomial, is established through the so-called Green's function, defined by

$$G(z,\tau) \equiv \frac{1}{N}\left\langle \mathrm{Tr}\left[z - H(\tau)\right]^{-1}\right\rangle\,. \tag{24}$$

Note that when N goes to infinity, the poles of this function merge, forming a cut on the complex plane. The link is made with the well-known Sokhotski–Plemelj formula $\rho(\lambda) = \frac{1}{\pi}\lim_{\epsilon\to 0_\pm} \mathrm{Im} G(\lambda \mp i\epsilon)$ and the relation

$$G(z,\tau) = \lim_{N\to\infty} \frac{1}{N}\partial_z \ln U_N(z,\tau)\,. \tag{25}$$

Note that $f_N(z,\tau) \equiv \frac{1}{N}\partial_z \ln U_N(z,\tau)$ is the famous Cole–Hopf transform [16]. One can easily verify that the diffusion equation derived for the ACP in the previous section corresponds to the following Burgers equation for $f_N(z,\tau)$

$$\partial_\tau f_N(z,\tau) + f_N(z,\tau)\partial_z f_N(z,\tau) = -\frac{1}{2N}\partial_z^2 f_N(z,\tau)\,, \tag{26}$$

in which the "spatial" variable z is complex and the role of "viscosity" is played by $-1/2N$, a negative number. In the large N limit, the viscosity vanishes, $f_N(z,\tau) \to G(z,\tau)$ and equation (26) becomes the inviscid Burgers equation

$$\partial_\tau G(z,\tau) + G(z,\tau)\partial_z G(z,\tau) = 0\,. \tag{27}$$

This equation may be solved by determining the characteristic lines, curves labeled ξ along which the solution is constant, that is $G(z,\tau) = G_0(\xi)$, where $G_0(z) \equiv G(z,\tau=0)$. In this particular case, these characteristic lines are given in the (z,τ) hyperplane by the equation

$$z = \xi + \tau G_0(\xi)\,, \tag{28}$$

which can be solved given the initial condition $G_0(\xi)$. The characteristic lines are tangent to the so-called caustics, whose location is given by the condition

$$0 = \left.\frac{dz}{d\xi}\right|_{\xi=\xi_c} = 1 + \tau G_0'(\xi_c)\,. \tag{29}$$

Along the caustics, the mapping between z and ξ ceases to be one to one, and the characteristic method loses its validity.

We consider generic initial conditions of two types: $H(\tau=0) = 0$ and $H(\tau=0) = \mathrm{diag}(-a\ldots, a\ldots)$. The first one corresponds to $G_0 = \frac{1}{z}$, the second (for which we assume that N is even) amounts to $G_0 = \frac{1}{2(z-a)} + \frac{1}{2(z+a)}$. In the former scenario, for an infinitely large matrix, the spectrum forms a single connected interval throughout its evolution. In the latter, it initially occupies two separate domains of the real axis which, in turn, merge at some critical space-time point. Figure 1 pictures the time evolution of the two corresponding spectral densities. In both cases, the characteristic lines that are real at $\tau = 0$, remain on the plane $\mathrm{Im}z = 0$ throughout the time evolution — we depict them in Fig. 2. Those that are complex, on the other hand, are symmetric under complex conjugation and never cross each other until some time, when they hit the real line and end on the cut of the Green's function. The caustics live on the plane $\mathrm{Im}z = 0$ so long as they do not merge. Moreover, they move along the branching points of the resulting Green's function and mark, therefore, the edges of the spectra. Additionally, they constitute the positions of the shocks, curves in the (z,τ) space along which the characteristic lines have to be cut to ensure unambiguity of the solution for $G(z,\tau)$. Note finally, that if the complex characteristic lines were allowed to cross the cuts of the complex plane, they would form (in the second scenario) complex caustics evolving out of the merging point of the real ones. These are depicted by dashed lines in Fig. 2.

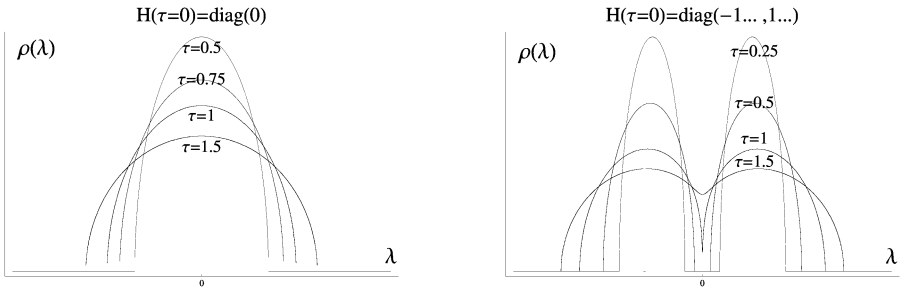

Fig. 1. The time evolution of the large N spectral density of the evolving matrices for two scenarios that differ in the imposed initial condition. The parameter a was set to one.

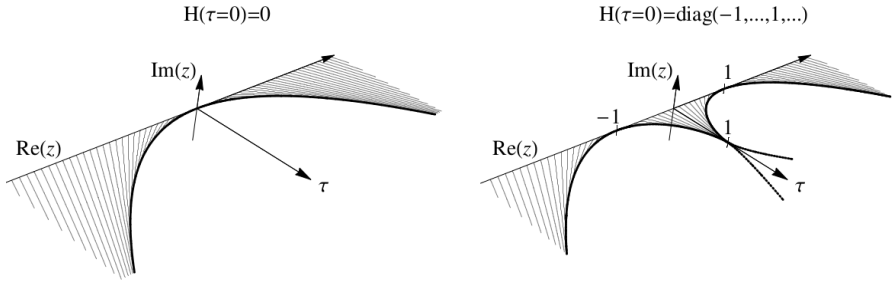

Fig. 2. The thin lines are characteristics that remain real throughout their temporal evolution. They finish at the bold lines which are caustics and shocks simultaneously. The dashed bold lines are the caustics that would be formed by the strictly complex characteristics (not depicted here) if they did not end on the branch cut.

5. Universal microscopic scaling

In this section, we inspect the ACP and the AICP in the vicinity of the points corresponding to the edges of the spectrum of $H(\tau)$, that is near the shocks. For the size of the matrix approaching infinity, one expects that the behaviors of the ACP and the AICP do not depend on the details of the stochastic process governing the evolution — a manifestation of the so-called microscopic universality.

As some of us have demonstrated [1], the asymptotic behavior of the ACP can be recovered by analyzing Eq. (26) through an expansion of f_N around the positions of the shocks. Following this approach, one could recover the Airy function describing the behavior of the ACP near the propagating edge. However, this method does not seem to be so effective when one examines a situation when two shocks collide. As we shall now see, it is more convenient in this case to return to the diffusion equation and realize that, irrespective

of the initial condition, the integral representations of the ACP and AICP have the following generic structure

$$\int_{\Gamma} e^{Nf(p,z,\tau)}dp\,. \tag{30}$$

This is well-suited for a steepest descent analysis in the large N limit [18]. Moreover, the saddle point condition $\partial_p f(p,z,\tau)|_{p=p_i} = 0$ is equivalent to equation (28). For the ACP, that is, we have

$$f(p,z,\tau) = \frac{1}{N}\ln\left[U_0(-ip)\right] - \frac{1}{2\tau}(p-iz)^2\,, \tag{31}$$

where $U_0(-ip) \equiv U_N(-ip, \tau = 0)$ and we identify ξ with $-ip_i$. For the AICP,

$$f(p,z,\tau) = \frac{1}{N}\ln\left[E_0(p)\right] - \frac{1}{2\tau}(p-z)^2\,, \tag{32}$$

(notice that $G_0(p) = -\frac{1}{N}\partial_p \ln\left[E_0(p)\right]$ and $E_0(p) \equiv E_N(p, \tau = 0)$) with ξ identified as p_i. The fact that the labels of the characteristics and the saddle points are connected is clearly not coincidental. The viscid Burgers equation and the diffusion equation are equivalent through the Cole–Hopf transform. This induces an equivalence between the characteristics method used to solve the inviscid limit of the former and the saddle point method applied for the large N solution of the latter. Consider approaching a caustic in the (z,τ) (hyper-)plane. Through the equivalence just pointed out, the merging of characteristics implies the merging of two saddle points — this will be the scenario of the first example considered below. When the caustics merge, forming a cusp, three saddle points coalesce, which will be studied subsequently.

The final issue to resolve before engaging the calculations is the question of what precisely we mean by the "vicinity" of the edges. If the width of the studied interval remains constant or shrinks too slowly, as the number of the eigenvalues grows to infinity, we will deal with an infinite number of eigenvalues and most of them will not "feel" that they are "close" to edge. On the other hand, if the interval shrinks too fast, in the end there will not be any eigenvalues left inside the interval. This is a heuristic explanation of why the studied vicinity of the edge should have a span proportional to the average spacing of the eigenvalues near the shock. This quantity can be derived by inspecting the large N limit of the spectral density that can be obtained from the Green's function. To proceed, one expands G around ξ_c

$$G_0(\xi) = G_0(\xi_c) + (\xi-\xi_c)G_0'(\xi_c) + \tfrac{1}{2}(\xi-\xi_c)^2 G_0''(\xi_c) + \tfrac{1}{6}(\xi-\xi_c)^3 G_0'''(\xi_c) + \ldots \tag{33}$$

From equation (28), we have $G_0 = (z-\xi)/\tau$ and $G_0'(\xi_c) = -1/\tau$, so that

$$z - z_c = \frac{\tau}{k!}(\xi - \xi_c)^k G_0^{(k)}(\xi_c) + \dots , \tag{34}$$

where k in $G_0^{(k)}(\xi_c)$ indicates the power of the first after (G_0'), non-vanishing derivative of G_0 taken in ξ_c, for a given critical point. This leads to

$$G(z,\tau) \simeq G_0(\xi_c) + G_0'(\xi_c)\left[\frac{k!(z-z_c)}{\tau G_0^{(k)}(\xi_c)}\right]^{1/k} . \tag{35}$$

Now, let N_Δ be the average number of eigenvalues located in an interval of width Δ near ξ_c. We have

$$N_\Delta \sim N \int_{z_c}^{z_c+\Delta} (z-z_c)^{1/k} dz \sim N\Delta^{1+1/k} , \tag{36}$$

which for fixed N_Δ, implies that Δ, or equivalently the average eigenvalue spacing, should scale with N as $N^{-k/(1+k)}$.

As we shall perform the rest of the calculations using the saddle point method, let us finally show how the proper scalings arise in that framework. This will be done, after [19], through the condition for the merging of saddle points p_i. In this context, the deviation s from z_c has to scale with the size of the matrix in such a way that, when N grows to infinity, the value of the integrand is not concentrated at separate p_is but in a single point p_c (corresponding to the merging of the saddle points). This is equivalent to requiring that for such an s, the distance between the saddle points p_i is of the order of the width of the Gaussian functions arising from expanding $f(p)$ around the respective p_is in $\exp[Nf(p)]$. In particular, the condition

$$|p_i - p_n| \sim \left[Nf''(p_j)\right]^{-1/2} , \qquad i \neq n , \tag{37}$$

(with p_j being any of the saddle points merging) gives the relevant order of magnitude of s, that is N^α. In other words, by plugging the formulas for the saddle points into (37) and substituting z with $z_c + s$ one obtains $s \sim N^\alpha$, with α depending on the number of saddle points merging. We, therefore, set $z = z_c + N^\alpha \eta$ and η is of the order of one. This subsequently sets the scale for the distance probed by the deviation from p_c and so, in the same manner, the condition $|p_i - p_c| \sim N^\beta$ defines the substitution $p = p_c + N^\beta t$.

The connection between the saddle points and characteristic lines allows us to relate β and α through k. First, note that by definition, α and k are related through $\alpha(1+k) = -k$. Near the critical point, we see, on the other

hand, that $|p_i - p_c| \sim |\xi - \xi_c|$ (through the equivalence of the saddle point condition and the prescription for the characteristics). Using Eq. (34), we thus obtain $\beta = \alpha/k = -(1+\alpha)$, which can be used as a consistency check.

In the example of Subsection 3.2, the merging of the saddle points happens in a particular critical time τ_c and there exists a time scale of the order of N^γ for which, asymptotically p_i are not distinguishable. This exponent is calculated by expanding the condition for the merging of saddle points around the critical value $\tau = \tau_c + N^\gamma \kappa$.

5.1. Soft, Airy scaling

Let us start by considering the simplest initial condition, namely $U_N(z, \tau = 0) = z^N$, for which $H(\tau = 0)$ is filled with zeros

$$U_N(z,\tau) = (-i)^N \sqrt{\frac{N}{2\pi\tau}} \int_{-\infty}^{\infty} q^N \exp\left(-\frac{N}{2\tau}(q - iz)^2\right) dq\,. \tag{38}$$

In this setting, the two edges of the spectrum plainly move in the opposite directions along the real line (see Fig. 1). We conduct a large N steepest descent analysis. As a first step, we set $f = \ln q - \frac{1}{2\tau}(q - iz)^2$ and obtain the saddle point equation as

$$\tau = q(q - iz)\,.$$

This relation has the role analogous to the equation governing the characteristics, as was introduced in the analysis of the Burgers equation and described above. The positions of the saddle points are given by $q_\pm = \frac{1}{2}\left(iz \pm \sqrt{4\tau - z^2}\right)$. Their merging, at $q_c = i\sqrt{\tau}$, marks the locations of the spectral edges. For simplicity, we focus on just one of them, in particular the right edge, for which $z_c = 2\sqrt{\tau}$. It is easily verified that this points to the crossing of the characteristic lines.

The new contour, going through q_c is depicted (for $\tau = 1$) in the left plot of Fig. 3. The contour deformation is constrained by two conditions: (a) the real part of the function f reaches a maximum along the contour and (b) the imaginary part must obey the condition $\mathrm{Im} f = \mathrm{const}$. By imposing the latter, we guarantee the steepest descent of $\mathrm{Re} f$ upon integrating along the contour. These requirements fix uniquely the path marked in bold in the left plot of Fig. 3.

The scaling exponent α is equal to $-\frac{2}{3}$ and we have $\eta = (z - 2\sqrt{\tau})N^{2/3}$. Moreover, $\beta = -\frac{1}{3}$ and the change of variables in the integral is given by $t = (q - i\sqrt{\tau})N^{1/3}$. These scaling parameters were obtained from equation (37) which compares the spacing between the saddle points $q_\pm$ to the width of the

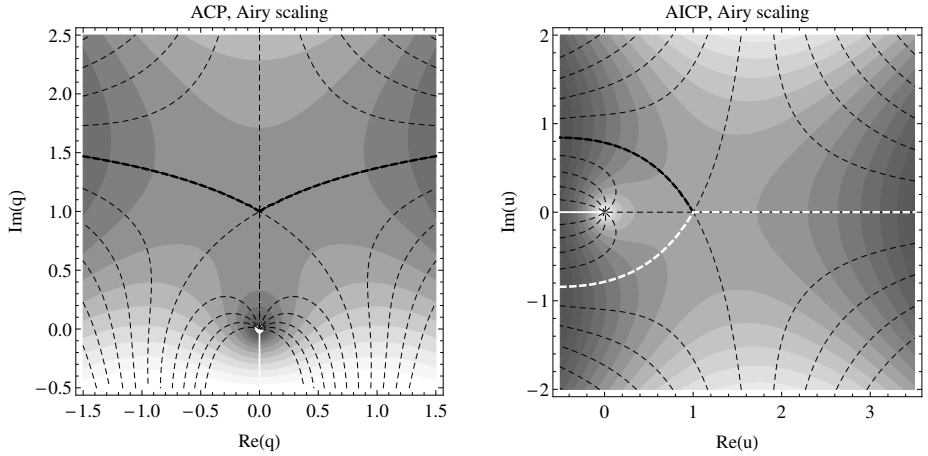

Fig. 3. The gray scale gradient in the graphs above portrays the value of $\mathrm{Re}f(p)$ (growing with the brightness), whereas the dashed lines depict the curves of constant $\mathrm{Im}f(p)$. The left figure is plotted for the ACP, with $p \equiv q$, the right one for the AICP, with $p \equiv u$. The initial condition is $H(\tau = 0) = 0$ and time τ is fixed to 1 for both. Dashed bold curves indicate contours of integration suitable for the saddle point analysis, for the AICP, we identify the black and white line with the contours Γ_+ and Γ_- respectively.

Gaussian approximation. The same result can be read out from the spectral density expanded around the edge of the bulk $\rho \sim \sqrt{|z - z_c|}$. Expanding the logarithm and taking the large matrix size limit yields

$$U_N\left(z = 2\sqrt{\tau} + \eta N^{-2/3}, \tau\right) \approx \tau^{N/2} \frac{N^{1/6}}{\sqrt{2\pi}} \exp\left(\frac{N}{2} + \frac{\eta N^{1/3}}{\sqrt{\tau}}\right) \mathrm{Ai}\left(\frac{\eta}{\sqrt{\tau}}\right), \tag{39}$$

where

$$\mathrm{Ai}(x) = \int_{\Gamma_0} dt \exp\left(\frac{it^3}{3} + itx\right) \tag{40}$$

is the well-known Airy function. The contour Γ_0 is formed by the rays $-\infty \times e^{5i\pi/6}$ and $\infty \times e^{i\pi/6}$ emerging as N goes to infinity. Along these rays, integral (40) is convergent as can be seen by substituting $t \to te^{i\pi/6}$.

In the case of the AICP, the initial condition used above takes the form of $E_N(z,\tau=0)=z^{-N}$ and (20) reads

$$E_N(z,\tau)=\sqrt{\frac{N}{2\pi\tau}}\int_{\Gamma_\pm} u^{-N}\exp\left(-N\frac{(u-z)^2}{2\tau}\right)du\,, \tag{41}$$

where the contours avoid the pole at zero from above Γ_+ (for $\mathrm{Im}z>0$) or below Γ_- (for $\mathrm{Im}z<0$), as explained in Subsection 3.2. This time, the saddle point merging for the right spectral edge occurs for $u_\mathrm{c}=\sqrt{\tau}$. It also marks the position of the cusps of the new integration contours (depicted for $\tau=1$ in the second plot of Fig. 3). The transformation of variables is given by $\eta=(z-2\sqrt{\tau})N^{2/3}$ and $it=(u-\sqrt{\tau})N^{1/3}$. Notice that the complex plane of t is rotated by $\pi/2$ with respect to the one of u. By expanding the logarithm and taking the large matrix size limit, we obtain

$$\begin{aligned}E_N\left(z=2\sqrt{\tau}+\eta N^{-2/3},\tau\right)\;\approx\;& i\tau^{-N/2}\frac{N^{1/6}}{\sqrt{2\pi}}\exp\left(-\frac{N}{2}-\frac{\eta N^{1/3}}{\sqrt{\tau}}\right)\\ &\times\mathrm{Ai}\left(e^{i\phi_\pm}\frac{\eta}{\sqrt{\tau}}\right)\,,\end{aligned} \tag{42}$$

the asymptotic behavior in terms of the Airy function, yet with its argument rotated by $\phi_+=-2\pi/3$, for $\mathrm{Im}z>0$, and by $\phi_-=2\pi/3$, for $\mathrm{Im}z<0$ in accordance with previous results for static matrices [20].

5.2. Pearcey scaling

To observe a collision of the edges of the spectrum, in the large N limit, one has to consider a slightly different initial condition. Let $U_N(z,\tau=0)=(z^2-a^2)^{N/2}$, with N even. This corresponds to the initial matrix eigenvalues set to $\pm a$ with equal degeneracy $N/2$. In this case, the ACP takes the form

$$U_N(z,\tau)=i^N\sqrt{\frac{N}{2\pi\tau}}\int_{-\infty}^{+\infty}dq\exp\left[-\frac{N}{2\tau}(q-iz)^2+\frac{N}{2}\log\left(a^2+q^2\right)\right]. \tag{43}$$

We determine the saddle point equation

$$\frac{q}{q^2+a^2}-\frac{q-iz}{\tau}=0 \tag{44}$$

and calculate its three solutions $q_{1,2,3}$. In this scenario, the two parts of the spectra join at $z_\mathrm{c}=0$ at $\tau_\mathrm{c}=a^2$ and this is reflected in the saddle points

merging for $q_c = 0$. The contour does not have to be deformed in this case as seen in the left plot of Fig. 4. Subsequently, the leading term of equation (37) relating the distance between the solutions $q_{1,2,3}$ to the Gaussian width of the saddle point approximation is used to extract the scaling $z \sim N^{-3/4}$. Further analysis of the distance between the critical saddle point q_c and the

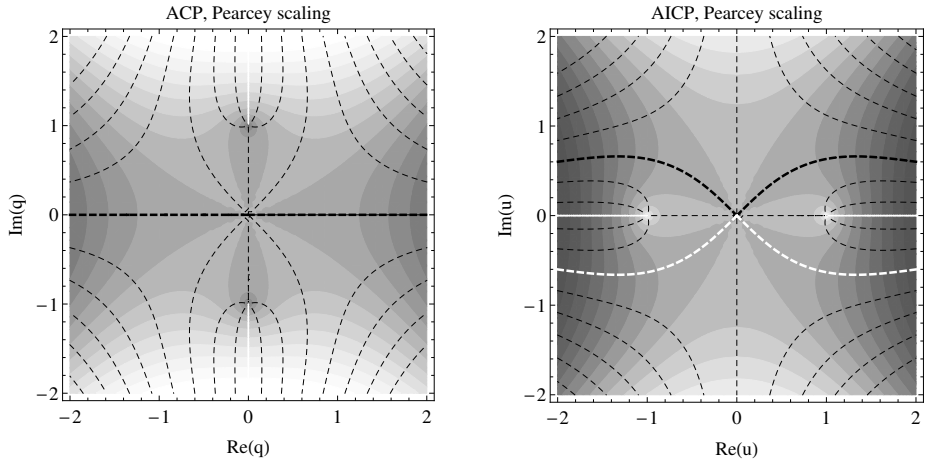

Fig. 4. In the two above graphs, the gray scale gradient portrays the value of $\mathrm{Re} f(p)$ (growing with the brightness), whereas the dashed lines depict the curves of constant $\mathrm{Im} f(p)$. The left figure is plotted for the ACP, with $p \equiv q$, the right one for the AICP, with $p \equiv u$. The initial condition is $H(\tau = 0) = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$ and time τ is fixed to 1 for both. Dashed bold curves indicate contours of integration suitable for the saddle point analysis, for the AICP, we identify the black and white line with the contours Γ_+ and Γ_- respectively.

points $q_{1,2,3}$ shows that the main contribution to the integral for $z \sim N^{-3/4}$ comes from q of the order of $N^{-1/4}$ and τ of the order of $N^{-1/2}$. Based on this, we set $q = tN^{-1/4}$, $\tau = a^2 + \kappa N^{-1/2}$, $z = \eta N^{-3/4}$. In the limit of $N \to \infty$, we expand the logarithm arising in the exponent through the initial condition and find

$$U_N\left(z = \eta N^{-3/4}, \tau \approx a^2 + \kappa N^{-1/2}\right) \approx \frac{N^{1/4}}{\sqrt{2\pi}} (ia)^N \mathrm{P}\left(\frac{\kappa}{2a^2}, \frac{\eta}{a}\right), \qquad (45)$$

where we define the Pearcey integral by

$$\mathrm{P}(x, y) = \int_{-\infty}^{\infty} dt \ \exp\left(-\frac{t^4}{4} + xt^2 + ity\right). \qquad (46)$$

This conclusion to the analysis of the microscopic behavior of the ACP is not a surprise — the Pearcey kernel was first derived [21] in the context of GUE matrices perturbed by a source [22], *i.e.* in a setting analogical to that considered here, in which the sources constitute the initial condition and their critical adjustment plays the role of the critical time.

In the case of the AICP, the initial condition reads $E_N(z,\tau=0) = (z^2-a^2)^{-N/2}$ and the solution to the complex diffusion equation is

$$E_N(z,\tau) = \sqrt{\frac{N}{2\pi\tau}} \int\limits_{\Gamma_\pm} du \left(u^2-a^2\right)^{-N/2} \exp\left(-\frac{N}{2\tau}(u-z)^2\right), \tag{47}$$

where $\Gamma_\pm$ denotes contours circling the poles $u_i = \pm a$ from above (Γ_+, $\mathrm{Im} z > 0$) or from below (Γ_-, $\mathrm{Im} z < 0$). The right plot in Fig. 4 depicts the types of curves the contours are deformed to. We parametrize the saddle point expansion by $\tau = a^2 + \kappa N^{-1/2}$, $z = \eta N^{-3/4}$ and $u = e^{i\pi/4} t N^{-1/4}$. We then obtain

$$\begin{aligned} E_N\left(z=\eta N^{-3/4}, \tau \approx a^2+\kappa N^{-1/2}\right) &\approx \frac{N^{1/4}}{\sqrt{2\pi}}(ia)^{-N} \\ \times \int\limits_{\Gamma_\pm} dt \ \exp&\left(-t^4/4 - \frac{i\kappa}{2a^2}t^2 + it\frac{e^{-i\pi/4}\eta}{a}\right), \end{aligned} \tag{48}$$

that is, a Pearcey type integral along contours Γ_+, Γ_- depending on the choice of sign of the imaginary part of z. The former is defined by rays with phase $\pi/2$ and 0 whereas the latter starts at $-\infty$ and after reaching zero forms a ray along a phase $-\pi/2$.

6. Constructing the kernel

The ACP and the AICP are the building blocks of the matrix kernel, which, in turn, contains, for arbitrary N, all the information about the matrix model. Obtaining the kernel was the aim of many previous works [10–12]. By making use of results of this paper, we can easily write down its form for the case of the studied diffusing matrix. We have

$$K_N(x,y,\tau) = \sum_{i=0}^{N-1} \Theta_i(x,\tau)\Pi_i(y,\tau), \tag{49}$$

where $\Theta_i(x,\tau)$ and $\Pi_i(y,\tau)$ are defined as follows. First, let

$$\begin{aligned}\Theta_{\vec{m}}(x,\tau) &\equiv E^+_{|\vec{m}|}(x,\tau) - E^-_{|\vec{m}|}(x,\tau) = \sqrt{\frac{N}{2\pi\tau}} \oint_{\Gamma_0} du \\ &\quad \times \exp\left(-N\frac{(u-x)^2}{2\tau}\right) E_0(u)\,, \end{aligned} \tag{50}$$

$$\Pi_{\vec{m}}(x,\tau) \equiv U_{|\vec{m}|}(x,\tau) = \sqrt{\frac{N}{2\pi\tau}} \int_{-\infty}^{\infty} dq \exp\left(-N\frac{(q-ix)^2}{2\tau}\right) U_0(-iq)\,. \tag{51}$$

Here, $E_0(x) = \prod_{i=1}^d (x-a_i)^{-m_i}$, $U_0(x) = \prod_{i=1}^d (x-a_i)^{m_i}$ are the initial conditions and $\vec{m}$ is a corresponding arbitrary eigenvalue multiplicity vector. Moreover, the contour Γ_0 encircles all the sources a_i clockwise and $E^+(z,\tau)$, $E^-(z,\tau)$ denote the different solutions of AICP diffusion equation valid for $\mathrm{Im}z > 0$ and $\mathrm{Im}z < 0$ respectively. Finally, the functions labeled by the index i in (49) arise through an ordering of the multiplicities according to their increasing norm $|\vec{m}| \equiv \sum_{j=1}^d m_j$ (see (A.6))

$$\Theta_i \equiv \Theta_{\vec{n}^{(i+1)}}\,, \qquad \Pi_i \equiv \Pi_{\vec{n}^{(i)}}\,, \qquad i = 0, \ldots, N-1\,. \tag{52}$$

The details concerning the construction of (49) are delivered in Appendix. It is shown in this appendix that the kernel for the Pearcey process that derives from the present construction is identical to that obtained by Brezin and Hikami [21]

$$\begin{aligned} K_{\mathrm{BH}}(x,y,\tau) &= -\frac{N}{2\pi\tau} \oint_{\Gamma_0} du \int_{-\infty}^{\infty} dq \frac{\left(-q^2-a^2\right)^{N/2}}{\left(u^2-a^2\right)^{N/2}} \frac{1}{u+iq} \\ &\quad \times \exp\left(-N\frac{(q-iy)^2}{2\tau} - N\frac{(u-x)^2}{2\tau}\right)\,. \end{aligned} \tag{53}$$

7. Conclusions

In this work, we have studied the behavior of averaged characteristic polynomials and averaged inverse characteristic polynomials associated with Hermitian matrices filled with entries performing Brownian motion in the space of complex numbers. A key new step of our analysis was a derivation of partial differential equations governing the matrix-valued evolution independently of the initial conditions. These turned out to be complex heat equations with diffusion coefficients inversely proportional to the size

of the matrices, and thus provided us with integral representations of the polynomials. By using the saddle point method, we were able to examine their so-called critical microscopic behavior, which is known to be universal. In particular, the asymptotics are driven by the Airy functions, at the edges of the spectrum, and by the Pearcey functions, when those edges meet. The first case holds for any frozen moment of time, however, one can easily modify the free diffusion into an Ornstein–Uhlenbeck problem to obtain the Airy behavior in the stationary limit at large time.

It is worth mentioning that the Pearcey function type behavior of the ACP arises also in the model of multiplicatively diffusing unitary matrices [23]. Moreover, the associated critical point marks the universality class of the Durhuus–Olesen type transition of Wilson loops in the Yang–Mills theory, in the limit of infinite number of colors [24–26]. Interestingly, our approach allowed us to recover the microscopic, critical behavior of the ACP for a diffusing chiral matrix. The resulting Bessoid function, an axially symmetric version of the Pearcey, is conjectured to describe the partition function of Euclidean QCD at the moment of spontaneous chiral symmetry breaking [27]. Finally, we note that an analogous, yet more intricate Burgers-like picture, arises also in the case of diffusing non-Hermitian matrices [28, 29], leading to novel, duality-type relations [30].

The authors are grateful to anonymous referee for valuable remarks and for pointing some relevant references. M.A.N. appreciates inspiring discussions with Nicholas Witte. P.W. is supported by the International Ph.D. Projects Programme of the Foundation for Polish Science within the European Regional Development Fund of the European Union, agreement No. MPD/2009/6 and the ETIUDA scholarship under the agreement No. UMO-2013/08/T/ST2/00105 of the Polish National Science Centre. M.A.N. and J.G. are supported by the Grant DEC-2011/02/A/ST1/00119 of the Polish National Science Centre.

Appendix

The kernel structure

To obtain (49), the formula for the kernel, we first present the connection between the diffusive model considered in this paper and the matrix model with a source introduced in [22]. First, let us notice that at time τ, the ensemble of the diffusing matrices H is equivalent to the ensemble of matrices defined by

$$X_\tau = H_0 + X\sqrt{\tau}\,, \tag{A.1}$$

where H_0 is a fixed matrix and X is random and given by the complex ($\beta = 2$) GUE measure

$$P(X)dX \sim \exp\left(-\frac{N}{2}\operatorname{Tr} X^2\right) dX\,. \tag{A.2}$$

Since (A.1) is a linear transformation, we have $X = \frac{X_\tau - H_0}{\sqrt{\tau}}$ and $dX \sim dX_\tau$. We, therefore, recover a matrix model with a source in which τ is just a parameter

$$P(X_\tau)dX_\tau \sim \exp\left(-\frac{N}{2\tau}\operatorname{Tr}(X_\tau - H_0)^2\right) dX_\tau\,. \tag{A.3}$$

The matrix H_0, corresponding in our formalism to the initial condition at $\tau = 0$, is the source matrix. From now on, we follow closely the works on random matrices with a source [10–12]. The matrix H_0 can be written in a diagonal form as

$$H_0 = \operatorname{diag}\left(\underbrace{a_1\ a_1\ \ldots}_{n_1};\underbrace{a_2\ a_2\ \ldots}_{n_2};\ldots;\underbrace{a_d\ \ldots}_{n_d}\right)\,,$$

with d eigenvalues a_i of multiplicities n_i. Out of the degeneracies, we form a vector $\vec{n} = (n_1, \ldots, n_d)$ which has a norm $|\vec{n}| \equiv \sum_{i=1}^{d} n_i = N$ dictated by the matrix size. We subsequently introduce, after [12], the multiple orthogonal polynomials of type I and II. The functions of type I are defined on the real line through

$$\begin{aligned}\Theta_{\vec{m}}(x,\tau) \;\equiv\; E^{+}_{|\vec{m}|}(x,\tau) - E^{-}_{|\vec{m}|}(x,\tau) &= \sqrt{\frac{N}{2\pi\tau}} \oint_{\Gamma_0} du \\ &\times \exp\left(-N\frac{(u-x)^2}{2\tau}\right) E_{0,\vec{m}}(u)\,,\end{aligned} \tag{A.4}$$

with an arbitrary multiplicity vector $\vec{m}$, an initial condition $E_{0,\vec{m}}(x) = \prod_{i=1}^{d}(x-a_i)^{-m_i}$ and the contour Γ_0 encircling all a_is clockwise. This contour arose since the AICP was defined in (20) by two different contours Γ_- and Γ_+ for $\mathrm{Im} z < 0$ and $\mathrm{Im} z > 0$ respectively.

Analogously, polynomials of type II are defined through the averaged characteristic polynomial by

$$\Pi_{\vec{m}}(x,\tau) \equiv U_{|\vec{m}|}(x,\tau) = \sqrt{\frac{N}{2\pi\tau}} \int_{-\infty}^{\infty} dq\ \exp\left(-N\frac{(q-ix)^2}{2\tau}\right) U_{0,\vec{m}}(-iq)\,, \tag{A.5}$$

with an initial condition $U_{0,\vec{m}}(x) = \prod_{i=1}^{d}(x-a_i)^{m_i}$. We stress the dependency of the polynomials on the multiplicity vector $\vec{m}$ of arbitrary norm $|\vec{m}| \neq N$. As a last step, we introduce an ordering of the vector $\vec{n}$

$$\begin{aligned}
\vec{n}^{(0)} &= (0,0,\dots,0)\,,\\
\vec{n}^{(1)} &= (1,0,\dots,0)\,,\\
&\vdots\\
\vec{n}^{(n_1)} &= (n_1,0,\dots,0)\,,\\
\vec{n}^{(n_1+1)} &= (n_1,1,\dots,0)\,,\\
&\vdots\\
\vec{n}^{(N)} &= (n_1,n_2,\dots,n_d)\,,
\end{aligned} \tag{A.6}$$

which forms a "nested" sequence increasing in norm. This sequence is exploited to compose N pairs of type I and type II polynomials

$$\Theta_i \equiv \Theta_{\vec{n}^{(i+1)}}\,, \qquad \Pi_i \equiv \Pi_{\vec{n}^{(i)}}\,, \qquad i = 0,\dots,N-1\,. \tag{A.7}$$

This stack of functions forms a kernel valid for an arbitrary source H_0

$$K_N(x,y) = \sum_{i=0}^{N-1} \Theta_i(x)\Pi_i(y)\,. \tag{A.8}$$

As an example, we consider the case of $a_1 = a, a_2 = -a$ and multiplicities $n_1 = n_2 = N/2$. We plug in the integral representations (A.4) and (A.5)

$$K_N(x,y) = \frac{N}{2\pi\tau}\oint_{\Gamma_0} du \int_{-\infty}^{\infty} dq \exp\left(-N\frac{(q-iy)^2}{2\tau} - N\frac{(u-x)^2}{2\tau}\right) I(q,u)\,, \tag{A.9}$$

where the sum over the initial conditions is denoted by $I(q,u)$. In our example, it is equal to

$$\begin{aligned}
I(q,u) &= \sum_{j=0}^{\frac{N}{2}-1} \frac{(-iq-a)^j}{(u-a)^{j+1}} + \frac{(-iq-a)^{N/2}}{(u-a)^{N/2}} \sum_{j=0}^{\frac{N}{2}-1} \frac{(-iq+a)^j}{(u+a)^{j+1}}\\
&= \frac{1}{u+iq}\left(1 - \frac{\left(-q^2-a^2\right)^{N/2}}{\left(u^2-a^2\right)^{N/2}}\right)\,.
\end{aligned}$$

By noticing that, under the integral, the first term vanishes, we arrive at the formula (53) given by Brezin and Hikami [21].

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IOP Publishing Journal of Physics A: Mathematical and Theoretical
J. Phys. A: Math. Theor. **49** (2016) 015201 (18pp) doi:10.1088/1751-8113/49/1/015201

Diffusion method in random matrix theory

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Received 23 June 2015, revised 19 October 2015
Accepted for publication 28 October 2015
Published 24 November 2015

Abstract

We introduce a calculational tool useful in computing ratios and products of characteristic polynomials averaged over Gaussian measures with an external source. The method is based on Dyson's Brownian motion and Grassmann/complex integration formulas for determinants. The resulting formulas are exact for finite matrix size N and form integral representations convenient for large N asymptotics. Quantities obtained by the method are interpreted as averages over standard matrix models. We provide several explicit and novel calculations with special emphasis on the $\beta = 2$ Girko-Ginibre ensembles.

Keywords: random matrix theory, characteristic polynomials, diffusion equation

1. Introduction

One of the strengths of random matrix theory lies in the abundance of calculational tools, with the method of orthogonal polynomials [1], supersymmetric techniques [2, 3], and free probability [4, 5] among many others. This paper attempts to enlarge this family with a technique we call the diffusion method. It serves as a framework for dealing with the powers and ratios of characteristic polynomials averaged over Gaussian measures with an external source. It began as a byproduct of considerations in quantum chromodynamics (hereafter QCD) made several years ago [6] and was thereafter successfully applied to Hermitian, Wishart and chiral models [7–9]. The method uses a Dyson-like picture of dynamical matrices and Grassmann/complex integral representation of determinants.

Studying characteristic polynomials in the random matrix theory (hereafter RMT) community is now a prolific topic with many branches, but its root can be traced back to a remarkable formula relating a characteristic polynomial averaged over a $\beta = 2$ Gaussian ensemble or Gaussian unitary ensemble (hereafter GUE) to a corresponding orthogonal polynomial:

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$$C_N^{-1}\int dH e^{-\mathrm{Tr}\,H^2}\det(z-H)=\pi_N(z),$$

where H is an $N\times N$ Hermitian matrix, $C_N=\int dH e^{-\mathrm{Tr}\,H^2}$ is a normalization constant, and $\pi_N(z)$ is a monic Hermite polynomial satisfying the orthogonality relation $\int dx e^{-x^2}\pi_n(x)\pi_m(x)=\delta_{nm}\sqrt{\pi}k!/2^k$. Such objects for $\beta=1,2$ Gaussian ensembles were considered in [10, 27] and in many areas of application such as zeroes of Riemann ζ function [12], eigenvalue statistics in quantum chaotic systems [13], and matrix models of QCD [14]. Moreover, products and ratios of characteristic polynomials reveal rich mathematical structures in both Gaussian orthogonal, and unitary and symplectic ensembles (corresponding to $\beta=1,2,4$ and hereafter abbreviated by GOE, GUE, GSE) [15–17] and $\beta=1,2$ Girko-Ginibre ensembles (abbreviated as GGE_β) [18–21, 37].

At the core of the method lies a seminal work of Dyson [22] who observed that a static matrix model of GOE, GUE, or GSE can also be interpreted as a dynamical system. He showed that the joint probability density function for N eigenvalues behaves exactly like a statistical system of N 'particles' interacting via the logarithmic potential. The system thus undergoes a Dysonian Brownian motion defined by the Langevin equation of the form:

$$d\lambda_i=\sum_{j(\neq i)}\frac{1}{\lambda_i-\lambda_j}dt+W_i dt,$$

where $W_i(t)$ is a delta-correlated, zero-mean Gaussian stochastic process $\left\langle W_i(t)W_j(t')\right\rangle=\delta_{ij}\delta(t-t')$. This Brownian motion of eigenvalues λ_i is induced by a Gaussian diffusion applied independently to all matrix entries. In this paper, this type of dynamics is called *entrywise* diffusion. Studying RMT from the Brownian motion's point of view has attracted attention of physicists [23, 24] and mathematicians [25, 26] alike.

The method is as discussed follows—we introduce *entrywise* dynamics to a matrix M and consider an averaged quantity of choice (i.e., the product or ratio of characteristic determinants but possibly others may apply) dependent on both M and parameter Λ_0. Then, upon proper deformation $\Lambda_0\to\Lambda$, we find a dual- diffusion equation of this quantity in the Λ-space, which is in turn solved easily. In the end, we perform the undeformed limit.

A phenomenon where the dynamics on M induces dual dynamics in some other parameters is generally known as duality and can be found when statistical quantities are characterized by two kinds of variables—random M over which the average is taken and fixed parameters Λ_0 (i.e., the argument z in the characteristic polynomial $\det(z-M)$). It is also a general feature of RMT models that these two groups are dual or interchangeable (i.e., averages over M with fixed Λ_0 can be related to averages over Λ_0 with fixed M [30, 38]).

The main advantage of this approach lies in the fact that the dual equation has considerably lower dimension and is solved readily by heat kernel techniques. It is also readily generalized to multi-matrix models (see section 3.4) and has a built-in external source matrix models.

This paper is organized as follows. In section 2 we discuss the method's framework—constructing the dual diffusion equation and the deformation parameters. We comment on the properties and limitations of the method and establish a relation to standard random matrix models. In section 3 we calculate five examples with special attention given to $\mathrm{GGE}_{\beta=2}$. We show how to arrive at the known formula for the ratio of characteristic polynomials averaged over GUE with an external source and derive a novel duality-type equation for averaged products of characteristic polynomials in the $\mathrm{GGE}_{\beta=2}$. Furthermore, we compute a new integral representation of the averaged characteristic polynomial in $\mathrm{GGE}_{\beta=2}$ with variance structure, compute the same object for a multiplication of two $\mathrm{GGE}_{\beta=2}$ matrices, and study a

$\mathrm{GGE}_{\beta=1}/\mathrm{GGE}_{\beta=2}$ crossover model. The examples obtained provided in the last section are mostly novel results.

2. Diffusion method

We introduce an *entrywise* diffusive dynamics to M—an $N \times N$ matrix of interest. Well-suited formalism for our purpose is the multidimensional heat equation:

$$\partial_\tau P(M, \tau) = \frac{1}{N}\Delta_M P(M, \tau), \tag{1}$$

where $P(M, \tau)$ is the joint probability density function, Δ_M denotes the Laplace operator over independent degrees of freedom of M, and the constant $1/N$ is a convention. For concreteness, we list Laplace operators realizing the canonical triad of GOE, GUE, and GSE:

$$\Delta_{GOE}^{\beta=1} = \sum_{i=1}^{N} \partial_{x_{ii}}^2 + \frac{1}{2}\sum_{\substack{i,j=1 \\ i<j}}^{N} \partial_{x_{ij}}^2, \qquad M_{kl} = x_{kl},\ x_{kl} = x_{lk}, \tag{2}$$

$$\Delta_{GUE}^{\beta=2} = \frac{1}{2}\sum_{i=1}^{N}\partial_{x_{ii}}^2 + \frac{1}{4}\sum_{\substack{i,j=1 \\ i<j}}^{N}\left(\partial_{x_{ij}}^2 + \partial_{y_{ij}}^2\right), \qquad M_{kl} = x_{kl} + iy_{kl},\ \begin{cases} x_{kl} = x_{lk} \\ y_{kl} = -y_{lk} \end{cases}, \tag{3}$$

$$\Delta_{GSE}^{\beta=4} = \frac{1}{4}\sum_{i=1}^{N}\partial_{x_{ii}}^2 + \frac{1}{8}\sum_{\substack{i,j=1 \\ i<j}}^{N}\left(\partial_{x_{ij}}^2 + \partial_{y_{ij}}^2 + \partial_{u_{ij}}^2 + \partial_{v_{ij}}^2\right),$$

$$M_{kl} = \begin{pmatrix} x_{kl} + iy_{kl} & u_{kl} + iv_{kl} \\ -u_{kl} + iv_{kl} & x_{kl} - iy_{kl} \end{pmatrix}, \begin{cases} x_{kl} = x_{lk} \\ y_{kl} = -y_{lk} \\ u_{kl} = -u_{lk} \\ v_{kl} = -v_{lk} \end{cases}, \tag{4}$$

where the symmetries arise from the Hermiticity condition $M = M^\dagger$. The family of $\mathrm{GGE}_{\beta=1,2,4}$ read:

$$\Delta_{GGE}^{\beta=1} = \frac{1}{4}\sum_{i,j=1}^{N}\partial_{x_{ij}}^2, \qquad M_{kl} = x_{kl}, \tag{5}$$

$$\Delta_{GGE}^{\beta=2} = \frac{1}{4}\sum_{i,j=1}^{N}\left(\partial_{x_{ij}}^2 + \partial_{y_{ij}}^2\right), \qquad M_{kl} = x_{kl} + iy_{kl}, \tag{6}$$

$$\Delta_{GGE}^{\beta=4} = \frac{1}{4}\sum_{i,j=1}^{N}\left(\partial_{x_{ij}}^2 + \partial_{y_{ij}}^2 + \partial_{u_{ij}}^2 + \partial_{v_{ij}}^2\right), \qquad M_{kl} = \begin{pmatrix} x_{kl} + iy_{kl} & u_{kl} + iv_{kl} \\ -u_{kl} + iv_{kl} & x_{kl} - iy_{kl} \end{pmatrix}. \tag{7}$$

All the instances of Δ_M are Gaussian, since in this work we do not consider non-Gaussian ensembles.

The objects of interest are the ratios and products of characteristic polynomials denoted as $D(Z, M)$. For example, we study in section 3.1 an object $D(Z, M) = \frac{\det(z - M)}{\det(w - M)}$ with $Z = \{z, w\}$. We are interested in formulas for the average

$$\overline{D_\tau}(Z) \coloneqq \left\langle D\left(Z, M_\tau\right)\right\rangle_{M_\tau} = C^{-1}\int dMP(M, \tau)D(Z, M), \tag{8}$$

where $\langle\rangle_{M_\tau}$ is a normalized averaging over the dynamical matrices M_τ and C is the normalization constant. The second equality is a consequence of the definition of the normalized joint probability density function $P(M, \tau) = \langle\delta(M_\tau - M)\rangle_{M_\tau}$.

To proceed, we extend $D(Z, M) \to D(Z, M; \Lambda)$ by introducing parameter-like variables Λ such that $\lim_{\Lambda\to\Lambda_0} D(Z, M; \Lambda) = D(Z, M)$ with $\Lambda_0 = 0$ in most cases. Even though the parameters Z are kept distinct from Λ, this division is purely conventional. At this point the deformation is defined in an abstract way but an algorithm for constructing Λ's is discussed in section 2.1. However, the purpose of this extension is clear—we search for a dual diffusive equation for the averaged deformed quantity $\overline{D_\tau}(Z; \Lambda) = \langle D(Z, M_\tau; \Lambda)\rangle_{M_\tau}$ in the Λ-parameter space.

In order to find it, we consider a time derivative of $\overline{D_\tau}$:

$$\partial_\tau\overline{D_\tau} = \frac{1}{N}\int dM\Delta_M P(M, \tau)D(Z, M; \Lambda) = \frac{1}{N}\int dMP(M, \tau)\Delta_M D(Z, M; \Lambda), \tag{9}$$

where we use equation (1) and integrated by parts to move the differential operator to D. Note that for Gaussian Δ_M (i.e., containing only second derivatives) integration by parts is tractable and does not produce any boundary terms for well-behaving functions P and D. The remaining task is to find $\mathbf{\Delta}_\Lambda$ such that the condition

$$\Delta_M D(Z, M; \Lambda) = \mathbf{\Delta}_\Lambda D(Z, M; \Lambda) \tag{10}$$

is satisfied. We then write the dual diffusive equation as

$$\partial_\tau\overline{D_\tau}(Z; \Lambda) = \frac{1}{N}\mathbf{\Delta}_\Lambda\overline{D_\tau}(Z; \Lambda). \tag{11}$$

As can be seen from condition (10), the Gaussian Laplace operators acting on the M manifold are transformed into Gaussian Laplace operators on the Λ space but, at the same time, we observe a decrease in the number of variables. This fact enables us to solve an initial value problem with a heat kernel K_τ:

$$\overline{D_\tau}(Z; \Lambda) = K_\tau\left(\Lambda, \Lambda'\right) \circ \overline{D_{\tau=0}}\left(Z; \Lambda'\right), \tag{12}$$

where '$\circ$' denotes a convolution operator and K_τ is defined by $\left(\partial_\tau - \frac{1}{N}\mathbf{\Delta}_\Lambda\right)K_\tau = 0$, $\lim_{\tau\to 0} K_\tau(\Lambda, \Lambda') = \delta(\Lambda - \Lambda')$. As a last step, the undeformed average is

$$\overline{D_\tau}(Z) = \lim_{\Lambda\to\Lambda_0} K_\tau\left(\Lambda, \Lambda'\right) \circ \overline{D_{\tau=0}}\left(Z; \Lambda'\right). \tag{13}$$

Concrete forms of K_τ are known once we specify the problem at hand.

2.1. Constructing Λ deformations

Until now we have described how to arrive at the diffusion equation (11) in the Λ-space. Now we turn to a procedure for finding a particular deformation Λ.

We start by opening the undeformed object $D(Z, M)$ with the use of the Grassmann/complex representation of determinants:

$$\det M \sim \int d\eta d\bar{\eta}\, \exp\left(\sum_{i,j=1}^{N}\bar{\eta}_i M_{ij}\eta_j\right), \qquad \frac{1}{\det M} \sim \int d\alpha\, \exp\left(\sum_{i,j=1}^{N}\bar{\alpha}_i M_{ij}\alpha_j\right), \tag{14}$$

where the proportionality constants are not essential in what follows. The variables η_i and α_i denote, respectively, Grassmann and complex sets of variables. Now suppose the undeformed

object D consists of k characteristic determinants and l inverse characteristic determinants; it is thus expressed as

$$D(Z, M) \sim \int d[\eta, \alpha] e^{T_G(Z,M;\eta,\alpha)}, \tag{15}$$

where T_G consists of k Grassmann and l complex binomials for every determinant and inverse determinant according to (14). A succinct notation for the measure reads $d[\eta, \alpha] = d\eta^{(1)} d\bar{\eta}^{(1)} \ldots d\eta^{(k)} d\bar{\eta}^{(k)} d\alpha^{(1)} \ldots d\alpha^{(l)}$.

With the help of (15), the action of the Laplacian Δ_M on $D(Z, M)$ is straightforward—it produces a certain polynomial U in both Grassmann and complex variables:

$$\Delta_M D(Z, M) \sim \int d[\eta, \alpha] U(\eta, \alpha) e^{T_G(Z,M;\eta,\alpha)}. \tag{16}$$

We assume that possible deformations Λ should not mix with the matrix M in the exponent T_G. Such an assumption is not restrictive since the structure of U already hints at particular types of deformations. However, now we observe that the action of Δ_M on a deformed $D(Z, M; \Lambda)$ should produce the same polynomial U albeit with a different exponent $T'_G = T_G + \delta T_G$:

$$\Delta_M D(Z, M; \Lambda) \sim \int d[\eta, \alpha] U(\eta, \alpha) e^{T'_G(Z,M;\Lambda;\eta,\alpha)}, \tag{17}$$

where δT_G is the unknown deformation part. To proceed, we now closely examine the structure of the polynomial U, which consists of terms with a general fourth-order structure:

$$a[v, w]_{nm} a\Big[v', w'\Big]_{n'm'}, \quad b[v, w]_{nm} b\Big[v', w'\Big]_{n'm'}, \quad c[v, w]_{nm} c\Big[v', w'\Big]_{n'm'} \tag{18}$$

where $a[v, w]_{nm} = \sum_{i=1}^{N} \bar{v}_i^{(n)} \bar{w}_i^{(m)}$, $b[v, w]_{nm} = \sum_{i=1}^{N} v_i^{(n)} w_i^{(m)}$, and $c[v, w]_{nm} = \sum_{i=1}^{N} \bar{v}_i^{(n)} w_i^{(m)}$ with variables v, w denoting either Grassmann η or complex α variables. The upper indices range over $n, m = 1 \ldots (k, l)$ and the choice of v, w is only restricted so that the whole term has even Grassmann variables (i.e., is of bosonic nature). The unknown deformation is therefore given by

$$\delta T_G = \sum_{v,w=\{\eta,\alpha\}} \sum_{m,n} \Big(\big(\lambda^a\big)_{mn} a[v, w]_{mn} + \big(\lambda^b\big)_{mn} b[v, w]_{mn} + \big(\lambda^c\big)_{mn} c[v, w]_{mn} \Big), \tag{19}$$

where the λ parameters need to be chosen such that the whole term is of bosonic nature (see the example in section 3.1 where the deformation parameters are fermionic in nature). This general form of δT_G is evident by observing that second-order differentiation wrt. λ's produce the fourth-order terms of type (18). Therefore, along with specifying δT_G, by such considerations we also construct the operator $\mathbf{\Delta}_\Lambda$. The choice of non-zero parameters λ in turn forms a deformation D that satisfies the condition (10) and so the averaged quantity satisfies a dual diffusion equation (11).

By considering many examples, we have found that only the terms of c-type are present in the $\beta = 2$ cases, whereas in the $\beta = 1, 4$ a, b-terms also form the polynomial U. To make this distinction explicit, we recall the definition of an undeformed D, which, after expanding the determinants, is also expressible as a large $(Nk + Nl) \times (Nk + Nl)$ superdeterminant of a diagonal supermatrix:

$$D \sim \mathrm{sdet}\Big[\mathrm{diag}\big(w_1 - M, w_2 - M, \ldots, w_l - M; z_1 - M, z_2 - M, \ldots z_k - M\big)\Big],$$

with the $Nk \times Nk$ fermionic-fermionic and $Nl \times Nl$ bosonic-bosonic blocks. In this interpretation, off-diagonal terms are expressible by c-type terms but a, b-type terms do not fit into this structure. This argument shows why we did not address the $\beta \neq 2$ cases—they are feasible but the results are harder to calculate since we lose determinantal structures on the dual side.

2.2. Relating diffusive dynamics to random matrix models

So far we have discussed a general framework in the diffusive language. Here we comment on how to connect this approach to static random matrix models usually considered in the RMT context. An *entrywise* diffusion (1) is, as a multidimensional heat equation, reinforced with an initial condition of a delta function type $(P^i)_\beta(M, \tau \to 0) = \delta(M - M_0)$. We thus solve it for the joint probability density function P with the Laplace operators given by (2)–(4):

$$\left(P^I\right)_\beta(M, \tau) = \left(C^I\right)_\beta^{-1} \exp\left(-\frac{N\beta}{4\tau}\,\mathrm{Tr}\left(M - M_0\right)^2\right), \tag{20}$$

where GOE ($\beta = 1$), GUE ($\beta = 2$), and GSE ($\beta = 4$) arise, respectively, and $(C^I)_\beta$ is the normalization constant. Likewise, plugging in the operators (5)–(7) of $\mathrm{GGE}_{\beta=1,2,4}$ forms the following joint probability density functions:

$$\left(P^{II}\right)_\beta(M, \tau) = \left(C^{II}\right)_\beta^{-1} \exp\left(-\frac{N}{\tau}\,\mathrm{Tr}\left(M - M_0\right)^\dagger\left(M - M_0\right)\right), \tag{21}$$

where $X^\dagger \to X^T$ for $\beta = 1$ and $X^\dagger \to ZX^TZ^T$ for $\beta = 4$, where $Z = \oplus_{i=1}^N \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $(C^{II})_\beta$ is the proportionality constant. Such random matrix models dependent on a fixed matrix M_0 are called models with an external source or shifted mean models [27]. Equivalently, the matrix at time τ is equal to

$$M_\tau = M_0 + \sqrt{\tau}\mathcal{M},$$

where $\mathcal{M}$ is a matrix chosen randomly from the respective joint probability density function $(P^i)_\beta(\mathcal{M}, \tau = 1; \mathcal{M}_0 = 0)$ at vanishing M_0 and fixed time $\tau = 1$. We thus conclude that averaging over dynamical matrices M_τ is equivalent to matrix models of variance proportional to τ with an external source M_0 applied.

2.3. General properties and resume

The method is applicable to general Gaussian *entrywise* diffusion (1) with examples given in equations (2)–(7). In addition to these canonical instances, in the example of section 3.3 we enlarge this family to include Gaussian diffusion with variance structure. A dual diffusion equation (11) in the parameter space has in general lower dimensionality when compared to the matrix size and is solved readily by heat kernel techniques. Because of the underlying diffusion process, the method has a built-in initial matrix M_0 translated into an external source considered in the standard random matrix models. The final formulas also can be viewed as integral representations convenient for large N analysis.

A general way to proceed follows these subsequent steps:

1. Introduce an *entrywise* diffusion of choice (1) (see the examples of (2)–(7))
2. Define object of interest D (i.e., product and ratios of determinants) and form a Λ parameter extension D according to section 2.1
3. Infer a diffusion equation in the Λ space for the averaged quantity $\overline{D}$ with the condition (10)
4. Solve the equation (11) using the heat kernel technique and set Λ parameters to its undeformed values Λ_0 to recover the object of interest

3. Examples

This section is devoted to several examples and serves as a tour-de-force showing the framework at work to calculate new results and compare to known ones. The majority of them deal with $\beta = 2$ Girko-Ginibre ensembles.

Example 1 is devoted to probably the most thoroughly studied Gaussian unitary ensemble. We show the applicability of our method to the averaged ratio of determinants, obtain an integral representation for any external source M_0, and show how it reduces to known results [16] for $M_0 \to 0$.

Example 2 elucidates on a certain duality-type formula for $\beta = 2$ Girko-Ginibre Ensemble, a result that continues the successful program of dualities obtained in both GUE [28, 30] and GGE [18].

Example 3 is a calculation of a $\beta = 2$ Girko-Ginibre ensemble with variance structure, a model considered in [31] and inspired by the doubly-correlated Wishart ensemble [32, 33]. We compute an integral representation and compare it to known results in the vanishing external source limit.

Example 4 serves as a proof-of-concept in applying the method to the multiplication of independent matrices drawn from the $\beta = 2$ Girko-Ginibre ensemble, which has attracted a lot of attention recently [34, 39, 40]. We calculate an integral representation for the averaged characteristic polynomial.

Our last example is a toy model used to study the crossover between $\beta = 1$ and a $\beta = 2$ Girko-Ginibre ensemble inspired by elliptic ensemble [41] modeling in a similar way as the GUE-GGE$_{\beta=2}$ transition. We arrive at the large N formula of the real-axis bump developed as we vary the crossover parameter.

3.1. Ratio of determinants for $\beta = 2$ Gaussian ensemble

In this example we calculate explicit formulas for the averaged ratio of determinants by the diffusion method for the GUE. For the Laplace operator of (3), an *entrywise* diffusion equation reads:

$$\partial_\tau P(M, \tau) = \frac{1}{2N}\left(\sum_{k=1}^{N}\partial_{x_{kk}}^2 + \frac{1}{2}\sum_{\substack{i,j=1\\ i>j}}^{N}\left(\partial_{x_{ij}}^2 + \partial_{y_{ij}}^2\right)\right)P(M, \tau),$$

where $M_{kl} = x_{kl} + iy_{kl}$ and $x_{kl} = x_{lk}$, $y_{kl} = -y_{lk}$. We consider the ratio of characteristic polynomials:

$$D(z,\, w,\, M) = \frac{\det(z - M)}{\det(w - M)}, \tag{22}$$

which is re-expressed using (14) as

$$\begin{aligned} D(z,\, w,\, M) &\sim \int d\,[\eta,\, \alpha] e^{T_G}, \\ T_G &= \sum_{k=1}^{N} \left(x_{kk} (\bar{\eta}_k \eta_k + \bar{\alpha}_k \alpha_k) - z \bar{\eta}_k \eta_k - w \bar{\alpha}_k \alpha_k \right) \\ &\quad + \sum_{\substack{k,l=1 \\ k<l}}^{N} x_{kl} (\bar{\eta}_k \eta_l - \eta_k \bar{\eta}_l + \bar{\alpha}_k \alpha_l + \alpha_k \bar{\alpha}_l) \\ &\quad + i \sum_{\substack{k,l=1 \\ k<l}}^{N} y_{kl} (\bar{\eta}_k \eta_l + \eta_k \bar{\eta}_l + \bar{\alpha}_k \alpha_l - \alpha_k \bar{\alpha}_l). \end{aligned}$$

We construct the deformation Λ following the steps given in section 2.1. First, the quantity $\Delta_M D(z,\, w,\, M)$ is calculated and the polynomial U of (16) is identified as

$$\begin{aligned} U &= \sum_{i=1}^{N} \bar{\eta}_i \eta_i \bar{\alpha}_i \alpha_i + \frac{1}{2} \sum_{i=1}^{N} \bar{\alpha}_i^2 \alpha_i^2 + \sum_{\substack{i,j=1 \\ i<j}}^{N} \left(\alpha_i \bar{\alpha}_j - \eta_i \bar{\eta}_j \right) \left(\bar{\eta}_i \eta_j + \bar{\alpha}_i \alpha_j \right) \\ &= - \sum_{\substack{i,j=1 \\ i<j}}^{N} \bar{\eta}_i \eta_i \bar{\eta}_j \eta_j + \frac{1}{2} \sum_{i=1}^{N} \bar{\alpha}_i^2 \alpha_i^2 + \sum_{\substack{i,j=1 \\ i<j}}^{N} \bar{\alpha}_i \alpha_i \bar{\alpha}_j \alpha_j + \sum_{\substack{i,j=1 \\ i<j}}^{N} \bar{\alpha}_i \eta_i \bar{\eta}_j \alpha_j \\ &\quad + \sum_{\substack{i,j=1 \\ i<j}}^{N} \bar{\eta}_i \alpha_i \bar{\alpha}_j \eta_j + \sum_{i=1}^{N} \bar{\eta}_i \eta_i \bar{\alpha}_i \alpha_i. \end{aligned}$$

The goal is to find a deformation parameter Λ and the corresponding Laplace operator reproducing this polynomial. As a first step, we calculate two derivatives wrt. parameters z and w:

$$\begin{aligned} \partial_{zz} D &\sim \int d\,[\eta,\, \alpha] \left(2 \sum_{\substack{i,j=1 \\ i<j}}^{N} \bar{\eta}_i \eta_i \bar{\eta}_j \eta_j \right) e^{T_G}, \\ \partial_{ww} D &\sim \int d\,[\eta,\, \alpha] \left(\sum_{i=1}^{N} \bar{\alpha}_i^2 \alpha_i^2 + 2 \sum_{\substack{i,j=1 \\ i<j}}^{N} \bar{\alpha}_i \alpha_i \bar{\alpha}_j \alpha_j \right) e^{T_G}, \end{aligned} \tag{23}$$

which already forms first three terms of U. To obtain the remaining ones we identify two a-type quantities $a\,[\eta,\, \alpha] = \sum_{i=1}^{N} \bar{\eta}_i \alpha_i$, $a'\,[\alpha,\, \eta] = \sum_{i=1}^{N} \bar{\alpha}_i \eta_i$ and thus establish two deformation parameters p and q forming δT_G:

$$\delta T_G = -\sum_{i=1}^{N} (\bar{\eta}_i p \alpha_i + \bar{\alpha}_i q \eta_i),$$

where the structure of (19) is evident and the chosen signs are a convention. The undeformed values of p, q are 0. Both are Grassmann numbers so that the δT_G is bosonic in nature. The deformed ratio $D(z,\, w,\, M;\, \Lambda)$ is

$$D(z,\, w,\, M;\, q,\, p) \sim \int d\,[\eta,\, \alpha] \exp\left[-\begin{pmatrix} \bar{\alpha} & \bar{\eta} \end{pmatrix} \begin{pmatrix} w - M & q \\ p & z - M \end{pmatrix} \begin{pmatrix} \alpha \\ \eta \end{pmatrix} \right]$$
$$= \int d\,[\eta,\, \alpha] e^{T'_G} = \mathrm{sdet} \begin{pmatrix} w - M & q \\ p & z - M \end{pmatrix},$$

where $T'_G = T_G + \delta T_G$ and the terms proportional to p and q form off-diagonal parts of the supermatrix. We calculate that

$$\partial_p \partial_q D \sim \int d\,[\eta,\, \alpha] \left(-\sum_{\substack{i,j=1 \\ i<j}}^{N} \bar{\alpha}_i \eta_i \bar{\eta}_j \alpha_j - \sum_{\substack{i,j=1 \\ i<j}}^{N} \bar{\eta}_i \alpha_i \bar{\alpha}_j \eta_j - \sum_{i=1}^{N} \bar{\eta}_i \eta_i \bar{\alpha}_i \alpha_i \right) e^{T'_G}$$

reproduces the remaining part of U and thus forms, together with (23), the Laplace operator in the parameter space:

$$\boldsymbol{\Delta}_\Lambda = \frac{1}{2} \left(\partial_{ww} - \partial_{zz} - 2\partial_p \partial_q \right).$$

The dual diffusion-like equation (11) is equal:

$$\partial_\tau \overline{D_\tau}(z,\, w;\, p,\, q) = \frac{1}{2N} \left(\partial_{ww} - \partial_{zz} - 2\partial_p \partial_q \right) \overline{D_\tau}(z,\, w;\, p,\, q). \tag{24}$$

We comment on two features of (24)—in the z direction it has a negative diffusivity constant and the diffusion also occurs in the p, q Grassmann 'directions.' In the RMT context the negative diffusive constant is interpreted as a source of a universal oscillatory behavior [9]. To deal with it on a technical level we can either Wick rotate the $z \to iz$ variable or consider instead a modified object $\frac{\det(iz - M)}{\det(w - M)}$, and we choose the former approach since it is more intuitive. In considering Grassmann 'diffusion' we make use of the well-known property of superdeterminants:

$$\mathrm{sdet} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{\det\left(d - ca^{-1}b \right)}{\det(a)}$$

and utilize the 'flatness' property $q^2 = 0$, $p^2 = 0$ to expand $D(z,\, w,\, H;\, p,\, q) = D^{(1)} + pD^{(2)} + qD^{(3)} + qpD^{(4)}$ in the Grassmann parameters. We rewrite (24) as an equivalent system of four equations for each $D^{(i)}$:

$$\partial_\tau \overline{D_\tau}^{(1)} = \frac{1}{2N} \left(\partial_{ww} - \partial_{zz} \right) \overline{D_\tau}^{(1)} - \frac{1}{N} \overline{D_\tau}^{(4)}, \tag{25}$$

$$\partial_\tau \overline{D_\tau}^{(2)} = \frac{1}{2N} \left(\partial_{ww} - \partial_{zz} \right) \overline{D_\tau}^{(2)}, \tag{26}$$

$$\partial_\tau \overline{D_\tau}^{(3)} = \frac{1}{2N} \left(\partial_{ww} - \partial_{zz} \right) \overline{D_\tau}^{(3)}, \tag{27}$$

$$\partial_\tau \overline{D_\tau}^{(4)} = \frac{1}{2N} \left(\partial_{ww} - \partial_{zz} \right) \overline{D_\tau}^{(4)}. \tag{28}$$

To find the solution of (24) we observe that only equations (25) and (28) contain relevant components $i = 1,\ 4$ since ultimately we are interested in the undeformed limit $\lim_{p,q \to 0} D = D^{(1)}$. To solve them we form a heat kernel of the Laplace operator $\frac{1}{2N}(\partial_{ww} - \partial_{zz})$:

$$K_\tau(z,\, w;\, y,\, v) = \frac{N}{2\pi\tau}\exp\left(-\frac{N}{2\tau}(v-w)^2 - \frac{N}{2\tau}(y-iz)^2\right), \tag{29}$$

where the z direction is Wick rotated, and we thus form the z dependent part of the solution to (25) and (28) by analytic continuation. The solution to (28) is

$$\overline{D_\tau}^{(4)}(z,\, w) = \int dydvK_\tau(z,\, w;\, y,\, v)D_0^{(4)}\big(-iy,\, v;\, M_0\big) =: \Big(K_\tau \circ D_0^{(4)}\Big)(z,\, w),$$

with M_0 denoting the initial matrix. With this notation, the solution to the inhomogeneous heat equation (25) is

$$\overline{D_\tau}^{(1)}(z,\, w) = \left(K_\tau \circ \left(D_0^{(1)} - \frac{\tau}{N}D_0^{(4)}\right)\right)(z,\, w) = \Big(K_\tau \circ D_0^{(1)}\Big)(z,\, w) - \frac{\tau}{N}\overline{D_\tau}^{(4)}(z,\, w). \tag{30}$$

To write it explicitly, we expand the initial condition:

$$D = \frac{\det\big(z - M_0\big)}{\det\big(w - M_0\big)}\left(1 + qp\ \mathrm{Tr}\,\frac{1}{\big(z - M_0\big)\big(w - M_0\big)}\right),$$

$$D_0^{(1)}\big(y,\, v;\, M_0\big) = \frac{\det\big(y - M_0\big)}{\det\big(v - M_0\big)},$$

$$D_0^{(4)}\big(y,\, v;\, M_0\big) = \frac{\det\big(y - M_0\big)}{\det\big(v - M_0\big)}\,\mathrm{Tr}\,\frac{1}{\big(y - M_0\big)\big(v - M_0\big)}.$$

Due to the unitary invariance, the most general initial matrix is diagonal $M_0 = \mathrm{diag}(h_1,\, \ldots,\, h_N)$, where some values of h_i can coincide. We also form an N-dimensional indexing vector $\vec{h} = (h_1,\, \ldots,\, h_N)$ and introduce two functions:

$$\pi_{\vec{h}}(z) = \sqrt{\frac{N}{2\pi\tau}}\int due^{-\frac{N}{2\tau}(u-iz)^2}\prod_{i=1}^{N}\big(-iu - h_i\big), \tag{31}$$

$$\theta_{\vec{h}}(w) = \sqrt{\frac{N}{2\pi\tau}}\int dqe^{-\frac{N}{2\tau}(q-w)^2}\prod_{i=1}^{N}\frac{1}{\big(q - h_i\big)}. \tag{32}$$

After setting $p,\, q \to 0$, the averaged ratio of characteristic polynomials (30) is equal to

$$\overline{D_\tau}(z,\, w) = \pi_{\vec{h}}(z)\theta_{\vec{h}}(w) - \frac{\tau}{N}\sum_{i=1}^{N}\pi_{\vec{h}_{-(i)}}(z)\theta_{\vec{h}_{+(i)}}(w), \tag{33}$$

where we introduced an extended $N+1$ dimensional vector $\vec{h}_+(i) = (h_1,\, \ldots,\, h_{i-1},\, h_i,\, h_i,\, h_{i+1},\, \ldots,\, h_N)$ and contracted $N-1$ dimensional vector $\vec{h}_-(i) = (h_1,\, \ldots,\, h_{i-1},\, h_{i+1},\, \ldots,\, h_N)$. To connect with known results, we write the average explicitly using (20) as

$$\overline{D_\tau}(z,\, w) = \big(C^I\big)_\beta^{-1}\int dHe^{-\frac{N}{2\tau}\mathrm{Tr}(M-M_0)^2)}\frac{\det(z - M)}{\det(w - M)}. \tag{34}$$

This quantity is present as a building block of biorthogonal structures [35, 36], where θ and π are the multiple orthogonal polynomials of type I and II, respectively.

To recover known formulas for the GUE case, we set $h_i = 0$ for all $i = 1 \ldots N$ so that $\vec{h}$, $\vec{h}_+(i)$ and $\vec{h}_-(i)$ become a N, $(N+1)$ and $N-1$ dimensional null vector, respectively. It is now more natural to introduce simplified notation: $\vec{h} \to N$, $\vec{h}_-(i) \to N-1$ and $\vec{h}_+(i) \to N+1$. Now the type I orthogonal polynomial associated with a $k \times k$ matrix is

given by $\theta_k(w) = \gamma_{k-1} f_{k-1}(w)$, where $\gamma_k = \frac{1}{k!}\left(\frac{N}{\tau}\right)^k \sqrt{\frac{N}{2\pi\tau}}$ and $f_k(z) = \int \frac{e^{-\frac{N}{2\tau}s^2}}{z-s}\pi_k(s)$ is the Cauchy transform. Along with $\gamma_N/\gamma_{N-1} = \frac{1}{\tau}$, we rewrite (33) as:

$$\overline{D_\tau}(z, w) = \gamma_{N-1}\pi_N(z) f_{N-1}(w) - \tau\gamma_N f_N(w)\pi_{N-1}(z),$$

which is the ratio formula calculated for GUE in [16].

3.2. Duality formula for $\beta = 2$ Girko-Ginibre ensemble

Let $M_{kl} = x_{kl} + iy_{kl}$ be an $N \times N$ matrix. We introduce an *entrywise* diffusive dynamics with a Laplacian (6):

$$\partial_\tau P(M, \tau) = \frac{1}{4N}\sum_{i,j=1}^{N}\left(\partial^2_{x_{ij}} + \partial^2_{y_{ij}}\right)P(M, \tau),$$

which also describes the $\mathrm{GGE}_{\beta=2}$. We aim at calculating an averaged product of k characteristic polynomials:

$$D^{(k)}(\mathcal{Z}, M) = \det\left[\prod_{i=1}^{k}\left(z_i - M\right)\left(\bar{z}_i - M^\dagger\right)\right]. \tag{35}$$

In this example we skip the procedure of constructing a deformed quantity $D^{(k)}$, which was described in section 2.1 and presented in the example of section 3.1. Deformation is a $2kN$ block matrix of the form

$$D^{(k)}(\mathcal{Z}, M; A) = \det\begin{pmatrix} \mathcal{Z}\otimes 1_N - 1_k \otimes M & -A^\dagger \otimes 1_N \\ A\otimes 1_N & \mathcal{Z}^\dagger \otimes 1_N - 1_k \otimes M^\dagger \end{pmatrix}, \tag{36}$$

where $\mathcal{Z} = \mathrm{diag}(z_1, \ldots z_k)$, 1_n is an N-dimensional unit matrix and A is a complex $k \times k$ matrix, representing the Λ-parameter space. We baptize $D^{(k)}$ the k-extended averaged characteristic polynomial (k-EACP) in agreement with [20] where the authors considered a particular case of $k = 1$. In the limit $\lim_{A\to 0} D^{(k)}$ we recover (35).

To proceed, we open the $D^{(k)}$ using Grassmann variables:

$$\begin{aligned} D^{(k)}(\mathcal{Z}, M; A) &\sim \int d[\eta, \xi]\ e^{T'_G}, \\ T'_G &= \sum_{i=1}^{k}\bar{\eta}^{(i)}\cdot\eta^{(i)} z_i + \sum_{i=1}^{k}\bar{\xi}^{(i)}\cdot\xi^{(i)}\bar{z}_i - \sum_{i=1}^{k}\bar{\eta}^{(i)}\cdot M\cdot\eta^{(i)} \\ &\quad - \sum_{i=1}^{k}\bar{\xi}^{(i)}\cdot M^\dagger\cdot\xi^{(i)} \\ &\quad \underline{- \sum_{i,j=1}^{k}\bar{\eta}^{(i)}\cdot\xi^{(j)}A^\dagger_{ij} + \sum_{i,j=1}^{k}\bar{\xi}^{(i)}\cdot\eta^{(j)}A_{ij}}, \end{aligned}$$

with kN-dimensional Grassmann vectors $\xi^{(i)}_j$ and $\eta^{(i)}_j$ $(i = 1 \ldots k, j = 1 \ldots N)$. We also introduced a dot '$\cdot$' denoting a sum over N-dimensional indices, a notation useful in this and forthcoming examples. The underlined part forms the deformation δT_G. We find that

$$\Delta_M D^{(k)} \sim \int d[\eta, \xi]\left(\sum_{i,j=1}^{k}\eta^{(i)}\cdot\bar{\xi}^{(j)}\bar{\eta}^{(i)}\cdot\xi^{(j)}\right)e^{T'_G},$$

which determines the parameter Laplace operator as

$$\mathbf{\Delta}_{\Lambda} = \frac{1}{4}\sum_{i,j=1}^{k}\left(\partial_{a_{ij}}^2 + \partial_{b_{ij}}^2\right) =: \operatorname{Tr}\partial_{AA^\dagger},$$

where $A_{kl} = a_{kl} + ib_{kl}$. We thus arrive at the final equation for the average $\langle D^{(k)}\rangle_{M_\tau} = \overline{D_\tau}^{(k)}(\mathcal{Z}; A)$:

$$\partial_\tau \overline{D_\tau}^{(k)}(\mathcal{Z}; A) = \frac{1}{N}\operatorname{Tr}\partial_{AA^\dagger}\overline{D_\tau}^{(k)}(\mathcal{Z}; A), \tag{37}$$

where we observe a dimensional reduction in diffusive variables $N \times N \to k \times k$. Using (21) and the proportionality constant $(C^{II})_2^{-1} = \left(\frac{N}{\pi\tau}\right)^{k^2}$, the solution is

$$\overline{D_\tau}^{(k)}(\mathcal{Z}; A) = \left(\frac{N}{\pi\tau}\right)^{k^2}\int dB e^{-\frac{N}{\tau}\operatorname{Tr}(B-A)\left(B^\dagger - A^\dagger\right)} D^{(k)}\left(\mathcal{Z}, M_0; B\right),$$

where A and M_0 are the initial values of the parameters- and the randomized matrix, respectively. We turn to the product of characteristic polynomials by taking the undeformed limit $A \to 0$:

$$\overline{D_\tau}^{(k)}(\mathcal{Z}) = \left(\frac{N}{\pi\tau}\right)^{k^2}\int dB e^{-\frac{N}{\tau}\operatorname{Tr} BB^\dagger} D^{(k)}\left(\mathcal{Z}, M_0; B\right). \tag{38}$$

To arrive at the duality formula, we write the definition of an average $\overline{D_\tau}^{(k)}$ using (21):

$$\overline{D_\tau}^{(k)}(\mathcal{Z}) = \left(\frac{N}{\pi\tau}\right)^{N^2}\int dM e^{-\frac{N}{\tau}\operatorname{Tr}(M-M_0)^\dagger(M-M_0)} D^{(k)}(\mathcal{Z}, M), \tag{39}$$

but this time $(C^{II})_2^{-1} = \left(\frac{N}{\pi\tau}\right)^{N^2}$. We can thus write the duality from (38) and (39):

$$\left(\frac{N}{\pi\tau}\right)^{N^2}\int dM e^{-\frac{N}{\tau}\operatorname{Tr} MM^\dagger} D^{(k)}\left(\mathcal{Z}, M + M_0; A = 0\right) = \left(\frac{N}{\pi\tau}\right)^{k^2}\int dB e^{-\frac{N}{\tau}\operatorname{Tr} BB^\dagger} D^{(k)}\left(\mathcal{Z}, M_0; B\right), \tag{40}$$

with the definition repeated for clarity:

$$D^{(k)}(\mathcal{Z}, M; A) = \det\begin{pmatrix} \mathcal{Z}\otimes 1_N - 1_k \otimes M & -A^\dagger \otimes 1_N \\ A\otimes 1_N & \mathcal{Z}^\dagger \otimes 1_N - 1_k \otimes M^\dagger \end{pmatrix}.$$

This new result is an extension of a similar formula for $M_0 = 0$ obtained in [18]. Such dual quantities were studied extensively in $\beta = 2$ Gaussian ensembles by [28], for general β in [30], and in the context of string theory by [29] among others.

3.3. $\beta = 2$ Girko-Ginibre ensemble with variance structure

In another example we deal with a $\mathrm{GGE}_{\beta=2}$ matrix model with variance structure. With $\tilde{M}_{kl} = \tilde{x}_{kl} + i\tilde{y}_{kl}$ we define it as

$$\partial_\tau \tilde{P}\left(\tilde{M}, \tau\right) = \frac{1}{4N}\sum_{i,j=1}^{N}\Gamma_{ii}^{-2}\Omega_{jj}^{-2}\left(\partial_{\tilde{x}_{ij}}^2 + \partial_{\tilde{y}_{ij}}^2\right)\tilde{P}\left(\tilde{M}, \tau\right),$$

where the variance structure is assumed to be strictly positive $\Gamma_{ii}, \Omega_{jj} > 0$. The fundamental solution is

$$\tilde{P}\big(\tilde{M},\,\tau\big)=\tilde{C}^{-1}\exp\Big(-\frac{N}{\tau}\,\mathrm{Tr}\,\Gamma^2\big(\tilde{M}-\tilde{M}_0\big)\Omega^2\big(\tilde{M}^\dagger-\tilde{M}_0^\dagger\big)\Big),$$

where $\tilde{M}_0$ denotes an initial matrix and $\tilde{C}^{-1}$ is a normalization constant. For $\tilde{M}_0=0$ this measure is called a doubly-correlated $\beta=2$ Wishart ensemble with both time (Γ) and space (Ω) correlations [33]. However, here we treat it as a Girko-Ginibre model (i.e., the eigenvalues of $\tilde{M}$ are investigated instead of the eigenvalues of $\tilde{M}^\dagger\tilde{M}$). A natural object of interest is a characteristic determinant:

$$\tilde{D}\big(z,\,\tilde{M}\big)=\det\big(z-\tilde{M}\big)\det\big(\bar{z}-\tilde{M}^\dagger\big). \tag{41}$$

It is convenient to consider a reparametrization $M=\Gamma\tilde{M}\Omega$, where the new matrix $M_{kl}=x_{kl}+iy_{kl}$ undergoes an usual *entrywise* diffusion equation (6):

$$\partial_\tau P(M,\,\tau)=\frac{1}{4N}\sum_{i,j=1}^{N}\Big(\partial^2_{x_{ij}}+\partial^2_{y_{ij}}\Big)P(M,\,\tau).$$

and the quantity of interest $\tilde{D}$ is modified to

$$\tilde{D}\big(z,\,\Gamma^{-1}M\Omega^{-1}\big)=D^{(\Gamma,\Omega)}(z,\,M)=\det\big(z-\Gamma^{-1}M\Omega^{-1}\big)\det\big(\bar{z}-\Omega^{-1}M^\dagger\Gamma^{-1}\big), \tag{42}$$

which we open using (14):

$$D^{(\Gamma,\Omega)}\sim\int d[\eta,\,\xi]e^{T_G},$$
$$T_G=\bar{\eta}\cdot\eta z+\bar{\xi}\cdot\xi\bar{z}-\bar{\eta}\cdot\big(\Gamma^{-1}X\Omega^{-1}\big)\cdot\eta-\bar{\xi}\cdot\Omega^{-1}M^\dagger\Gamma^{-1}\cdot\xi.$$

According to section 2.1, we look for a deformation by calculating the action of Laplacian $\Delta_M D^{(\Gamma,\Omega)}$ to obtain the polynomial U:

$$U=\sum_{i,j=1}^{N}\Gamma_{ii}^{-2}\xi_i\bar{\eta}_i\Omega_{jj}^{-2}\bar{\xi}_j\eta_j, \tag{43}$$

which depends on the variances Ω, Γ but the formula retains the structure of equation (18). Both variance matrices modify the c-type terms slightly:

$$c[v,\,w]=\sum_{i=1}^{N}\bar{v}_i w_i\to\sum_{i=1}^{N}V_{ii}\bar{v}_i w_i,\qquad V_{ii}=\big\{\Gamma_{ii},\,\Omega_{ii}\big\},$$

but we form the δT_G out of modified c-type terms as:

$$\delta T_G=-\bar{w}\bar{\eta}\cdot\Gamma^{-2}\cdot\xi+w\bar{\xi}\cdot\Omega^{-2}\cdot\eta,$$

with an introduced Λ-parameter w and arbitrary signs. By setting $T_G'=T_G+\delta T_G$, the deformed determinant $D^{(\Gamma,\Omega)}$ is expressible as a block matrix with off-diagonal elements encoding the deformation:

$$D^{(\Gamma,\Omega)}(z,\,M;\,w)=\det\begin{pmatrix}z-\Gamma^{-1}M\Omega^{-1} & -\Gamma^{-2}\bar{w}\\ \Omega^{-2}w & \bar{z}-\Omega^{-1}M^\dagger\Gamma^{-1}\end{pmatrix}, \tag{44}$$

with $\lim_{w\to 0}D^{(\Gamma,\Omega)}=D^{(\Gamma,\Omega)}$. In the last step, we find that the action of $\partial_{\bar{w}w}$ acting on $D^{(\Gamma,\Omega)}$ reproduces the polynomial (43) and thus forms Laplacian in the parameter space:

$$\boldsymbol{\Delta}_\Lambda=\partial_{w\bar{w}},$$

and the final equation for the averaged $\overline{D_\tau}$ is

$$\partial_\tau \overline{D_\tau}(z;\, w) = \frac{1}{N}\partial_{w\bar{w}}\overline{D_\tau}(z;\, w). \tag{45}$$

Because the resulting dual equation is two-dimensional, we readily form the solution:

$$\overline{D_\tau}(z) = \frac{N}{\pi\tau}\int d^2u e^{-\frac{N}{\tau}|u|^2} D^{(\Gamma,\Omega)}\big(z,\, M_0;\, u\big), \tag{46}$$

$$D^{(\Gamma,\Omega)}(z,\, M;\, w) = \det\begin{pmatrix} z - \Gamma^{-1}M\Omega^{-1} & -\Gamma^{-2}\bar{w} \\ \Omega^{-2}w & \bar{z} - \Omega^{-1}M^\dagger\Gamma^{-1}\end{pmatrix},$$

where we took the undeformed limit $w \to 0$. It is an integral representation valid for general M_0 and correlations Γ, Ω. For completeness, the averaged quantity $\overline{D_\tau}(z)$ is explicitly expressed with the use of the joint probability density function (21) as

$$\overline{D_\tau}(z) = (C^{II})_2^{-1}\int dM\, \exp\Big(-\frac{N}{\tau}\,\mathrm{Tr}\big(M - M_0\big)^\dagger\big(M - M_0\big)\Big) D^{(\Gamma,\Omega)}(z,\, M). \tag{47}$$

In the special $M_0 \to 0$ limit, the solution (46) reproduces the result of [31]:

$$\overline{D_\tau}(z) = \frac{2N}{\tau}\int_0^\infty d\rho\rho e^{-\frac{N}{\tau}\rho^2}\prod_{i=1}^N\Big(|z|^2 + \rho^2\Gamma_{ii}^{-2}\Omega_{ii}^{-2}\Big).$$

3.4. Multiplication of two independent $\beta = 2$ Girko-Ginibre matrices

In this example we show how the method is applied to a product of two $\beta = 2$ Girko-Ginibre matrices, a case that has drawn much attention recently [34, 39, 40]. We introduce two matrices, M_1, M_2, each undergoing an independent $\mathrm{GGE}_{\beta=2}$ *entrywise* diffusion of (6):

$$\begin{aligned}\partial_\tau P\big(M_1,\, M_2,\, \tau\big) = &\frac{1}{4N}\sum_{i,j=1}^N\Big(\partial_{x_{ij}}^2 + \partial_{y_{ij}}^2\Big)P\big(M_1,\, M_2,\, \tau\big)\\ &+ \frac{1}{4N}\sum_{i,j=1}^N\Big(\partial_{x'_{ij}}^2 + \partial_{y'_{ij}}^2\Big)P\big(M_1,\, M_2,\, \tau\big),\end{aligned}$$

where $(M_1)_{kl} = x_{kl} + iy_{kl}$ and $(M_2)_{kl} = x'_{kl} + iy'_{kl}$. We consider a determinant of the form:

$$D\big(z,\, M_1,\, M_2\big) = \det\big(z - M_1M_2\big)\det\Big(\bar{z} - \big(M_1M_2\big)^\dagger\Big). \tag{48}$$

To proceed, we linearize it by expanding the block structure accordingly:

$$D = \det\begin{pmatrix} z - M_1M_2 & 0 \\ 0 & \bar{z} - \big(M_1M_2\big)^\dagger\end{pmatrix} = \det\left(\begin{pmatrix} z & 0 \\ 0 & \bar{z}\end{pmatrix} - \begin{pmatrix} 0 & M_1 \\ M_2^\dagger & 0\end{pmatrix}\begin{pmatrix} 0 & M_1^\dagger \\ M_2 & 0\end{pmatrix}\right)$$

$$= \det\begin{pmatrix} z & 0 & 0 & X_1 \\ 0 & \bar{z} & M_2^\dagger & 0 \\ 0 & M_1^\dagger & 1 & 0 \\ M_2 & 0 & 0 & 1\end{pmatrix}$$

where we used formulas valid for block matrices:

$$\begin{pmatrix} M_1M_2 & 0 \\ 0 & (M_1M_2)^\dagger \end{pmatrix} = \begin{pmatrix} 0 & M_1 \\ M_2^\dagger & 0 \end{pmatrix}\begin{pmatrix} 0 & M_1^\dagger \\ M_2 & 0 \end{pmatrix},$$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det(ad - bc), \qquad \text{if } cd = dc.$$

After this preparatory transformation, we identify deformation parameters as described in section 2.1. We skip this part due to its similarity to previous examples and just write the resulting deformed characteristic polynomial:

$$D(z, M_1, M_2; u, v, w) = \det\begin{pmatrix} z & -\bar{w} & 0 & M_1 \\ v & \bar{z} & M_2^\dagger & 0 \\ 0 & M_1^\dagger & u & w \\ M_2 & 0 & -\bar{v} & \bar{u} \end{pmatrix}, \tag{49}$$

where three additional parameters u, v, w were introduced. We open this determinant:

$$\begin{aligned} D &\sim \int d\xi d\eta e^{T'_G}, \\ T'_G &= \bar{\xi}_1 \cdot \xi_1 z + \bar{\xi}_2 \cdot \xi_2 \bar{z} \\ &\quad + \underline{\bar{\eta}_1 \cdot \eta_1 u + \bar{\eta}_2 \cdot \eta_2 \bar{u} - \bar{\xi}_1 \cdot \xi_2 \bar{w} + \bar{\xi}_2 \cdot \xi_1 v + \bar{\eta}_1 \cdot \eta_2 w - \bar{\eta}_2 \cdot \eta_1 \bar{v}} \\ &\quad + \bar{\xi}_1 \cdot X_1 \cdot \eta_2 + \bar{\xi}_2 \cdot X_2^\dagger \cdot \eta_1 + \bar{\eta}_1 \cdot X_1^\dagger \cdot \xi_2 + \bar{\eta}_2 \cdot X_2 \cdot \xi_1, \end{aligned}$$

where ξ_i, η_i are Grassmann variables and the underlined part forms δT_G. The joint Laplace operator acting on D is

$$(\Delta_{M_1} + \Delta_{M_1})D \sim \int d[\eta, \xi](\bar{\xi}_1 \cdot \xi_2 \eta_2 \cdot \bar{\eta}_1 + \bar{\eta}_2 \cdot \eta_1 \xi_1 \cdot \bar{\xi}_2) e^{T'_G},$$

which also dictates the Laplace operator in the parameter space to be of the form

$$\boldsymbol{\Delta}_\Lambda = \partial_{w,\bar{w}} + \partial_{v,\bar{v}}.$$

The dual diffusion equation for the averaged determinant is

$$\partial_\tau \overline{D_\tau}(z; u, v, w) = \frac{1}{N}(\partial_{w,\bar{w}} + \partial_{v,\bar{v}})\overline{D_\tau}(z; u, v, w), \tag{50}$$

We write the solution in the undeformed limit $v, w \to 0$ and $u \to 1$:

$$\overline{D_\tau}(z) = \left(\frac{N}{\pi\tau}\right)^2 \int d^2w d^2v e^{-\frac{N}{\tau}(|w|^2+|v|^2)} D\left(z, (M_1)_0, (M_2)_0; 1, v, w\right). \tag{51}$$

As before, we investigate the vanishing source limit $(M_i)_0 \to 0$, where

$$D(z, 0, 0; u, v, w) = (1 + \bar{v}w)^N(|z|^2 + v\bar{w})^N.$$

The angles of w, v in (51) can be integrated out with the help of hypergeometric function ${}_2F_1$:

$$\overline{D_\tau}(z) = \left(\frac{2N}{\tau}\right)^2 \int_0^\infty dpdq\, qpe^{-\frac{N}{\tau}(q^2+p^2)}\left(|z|^2 + q^2p^2\right)^N {}_2F_1 \times \left(\frac{1-N}{2}, -\frac{N}{2}, 1, \frac{4\,|z|^2p^2q^2}{\left(|z|^2 + p^2q^2\right)^2}\right),$$

and we simplify it further by introducing $p^2 = t\alpha$, $q^2 = \frac{t}{\alpha}$ and integrating over α's:

$$\overline{D_\tau}(z) = \left(\frac{2N}{\tau}\right)^2 \int_0^\infty dt\; tK_0\left(\frac{2Nt}{\tau}\right)\left(|z|^2 + t^2\right)^N {}_2F_1\left(\frac{1-N}{2}, -\frac{N}{2}, 1, \frac{4\,|z|^2t^2}{\left(|z|^2 + t^2\right)^2}\right), \quad (52)$$

where K_0 is the modified Bessel function of the second kind. The average $\overline{D_\tau}$ is explicitly given as

$$\overline{D_\tau}(z) = (C^{II})_2^{-2} \int dM_1 dM_2 \exp\left(-\frac{N}{\tau}\,\mathrm{Tr}\left(M_1^\dagger M_1 + M_2^\dagger M_2\right)\right) \times \det\left(z - M_1M_2\right)\det\left(\bar{z} - \left(M_1M_2\right)^\dagger\right)$$

according to (21). To the best of our knowledge, this result has not been considered previously.

3.5. Girko-Ginibre ensemble crossover model between $\beta = 1$ and $\beta = 2$

The last example is a crossover model between real and complex Girko-Ginibre ensembles. A matrix drawn from $\mathrm{GGE}_{\beta=1}$ has either real or complex conjugated pairs of eigenvalues, whereas $\mathrm{GGE}_{\beta=2}$ is not constrained by such condition—its eigenvalues spread evenly over the complex plane. To study this transition, we introduce an *entrywise* diffusion combining the Laplace operators of (5) and (6):

$$\partial_\tau P(M, \tau) = \frac{1}{4N}\sum_{i,j=1}^{N}\left(\partial_{x_{ij}}^2 + \alpha^2\partial_{y_{ij}}^2\right)P(M, \tau),$$

which forms an $N \times N$ matrix $M_{kl} = x_{kl} + iy_{kl}$. The model introduces a crossover parameter α that varies between 0 ($\beta = 1$) and 1 ($\beta = 2$). We investigate the condensation of eigenvalues on the real line as we take the limit $\alpha \to 0$. We are interested in a standard characteristic polynomial:

$$D(z, M) = \det(z - M)\det\left(\bar{z} - M^\dagger\right). \quad (53)$$

After finding the deformation analogously to the examples of section 3.2 and 3.3, we form a deformed quantity:

$$D(z, M; w) = \begin{pmatrix} z - M & -\bar{w} \\ w & \bar{z} - M^\dagger \end{pmatrix}, \quad (54)$$

for which, using the same techniques as previously, we find a dual diffusion equation:

$$\partial_\tau \overline{D_\tau}(z; w) = \frac{1+\alpha^2}{2N}\partial_{w\bar{w}}\overline{D_\tau}(z; w). \quad (55)$$

The solution, after taking the w $\to$ 0 limit, is

$$\overline{D_\tau}(z) = \frac{2N}{\tau}\int_0^\infty dr r e^{-\frac{2N}{\tau(1+\alpha^2)}r^2} D(z, M_0; r), \tag{56}$$

which is valid for any external source M_0. For vanishing external source $M_0 \to 0$, the formula (56) agrees with the results for both $\mathrm{GGE}_{\beta=1,2}$ [19, 20].

We now turn to a microscopic crossover region of $\alpha \to 0$ and Im $z \to 0$, where a precursor of the real eigenvalues of $\mathrm{GGE}_{\beta=1}$ is visible. We set a microscopic scaling near the real axis $z = i\eta N^{-1/4}$ and the crossover parameter near zero $\alpha = aN^{-1/4}$, which yield an asymptotic formula:

$$\overline{D_\tau} \sim e^{-a^4/2} e^{-\frac{2a^2\eta^2}{\tau}} \operatorname{erfc}\left(\frac{\sqrt{2}\eta^2}{\tau} - \frac{a^2}{\sqrt{2}}\right).$$

It shows an error function type bump near $\eta = 0$, which we interpret as the discussed precursor of an emerging bulk of real eigenvalues.

4. Conclusions

The method presented here is applicable to the Gaussian random matrix models for all $\beta = 1, 2, 4$ and serves as a tool for obtaining averages of both ratios and products of characteristic polynomials. Its main goal is to find a dual diffusion equation in the parameter space when the matrix itself undergoes a similar diffusive motion.

We calculated several examples for $\beta = 2$ GUE and GGE, where the resulting dual diffusion equations were particularly simple. We found a novel duality formula for products of characteristic polynomials, for $\mathrm{GGE}_{\beta=2}$, a previously not considered characteristic polynomial for the product of two $\mathrm{GGE}_{\beta=2}$ matrices and a $\beta = 1/\beta = 2$ Girko-Ginibre ensemble crossover model. We also dealt with $\mathrm{GGE}_{\beta=2}$ with variance structure and re-derived the ratio of characteristic polynomials in GUE case.

The main advantage of this method is a large reduction in the degrees of freedom. It also has a built-in external source random matrix models, which is especially suitable when looking for duality formulas of type (40).

Acknowledgments

Author acknowledges the support of the Grant DEC-2011/02/A/ST1/00119 of the National Centre of Science and the Australian Government Endeavour Fellowship during which this work was done. He would also thank P J Forrester, M A Nowak and P Warchol for reading the manuscript and valuable comments.

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IOP Publishing Journal of Physics A: Mathematical and Theoretical
J. Phys. A: Math. Theor. **49** (2016) 085203 (26pp) doi:10.1088/1751-8113/49/8/085203

Hydrodynamical spectral evolution for random matrices

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Received 29 October 2015
Accepted for publication 3 November 2015
Published 18 January 2016

Abstract

The eigenvalues of the matrix structure $X + X^{(0)}$, where X is a random Gaussian Hermitian matrix and $X^{(0)}$ is non-random or random independent of X, are closely related to Dyson Brownian motion. Previous works have shown how an infinite hierarchy of equations satisfied by the dynamical correlations become triangular in the infinite density limit, and give rise to the complex Burgers equation for the Green's function of the corresponding one-point density function. We show how this and analogous partial differential equations, for chiral, circular and Jacobi versions of Dyson Brownian motion follow from a macroscopic hydrodynamical description involving the current density and continuity equation. The method of characteristics gives a systematic approach to solving the PDEs, and in the chiral case we show how this efficiently reclaims the characterization of the global eigenvalue density for non-central Wishart matrices due to Dozier and Silverstein. Collective variables provide another approach to deriving the complex Burgers equation in the Gaussian case, and we show that this approach applies equally as well to chiral matrices. We relate both the Gaussian and chiral cases to the asymptotics of matrix integrals.

Keywords: random matrix theory, Burgers equation, spectral density, Harish-Chandra–Itzykson–Zuber integral

1. Introduction

One of the most basic questions in random matrix theory asks for the limiting global spectral density, given the distribution on the elements of the matrices or the distribution on the space of matrices. Perhaps the most celebrated result in this class is the Wigner semi-circle law. It applies to real random symmetric, or complex Hermitian matrices, in which the diagonal

entries are independently distributed with mean zero and variance unity, and the upper triangular off-diagonal entries are independently distributed with mean zero and variance two. Random matrices of this type are referred to as Wigner matrices. Consider a specific class of Wigner matrices of size $N \times N$. A scaling of the matrices so that the eigenvalue density has compact support is referred to as a global limit. For Wigner matrices this is achieved by dividing each matrix by $\sqrt{2N}$ before taking the $N \to \infty$ limit. Moreover, the corresponding global spectral density $\rho^{\mathrm{Wig}}(x)$, normalized to integrate to unity, is given by

$$\rho^{\mathrm{Wig}}(x) = \begin{cases} \dfrac{2}{\pi}\sqrt{1-x^2}, & |x| \leqslant 1, \\ 0, & |x| > 1. \end{cases} \tag{1.1}$$

In its full generality, the most common proof of this result, and in fact the one provided by Wigner [1], is to show that

$$\lim_{N\to\infty} \left(\frac{\sqrt{2}}{N}\right)^{2k} \langle \mathrm{Tr}\, X^{2k} \rangle = \frac{1}{k+1}\binom{2k}{k}, \tag{1.2}$$

where the rhs is the kth Catalan number. This is equivalent to establishing that the moments $\int_{-1}^{1} x^{2k}\rho^{\mathrm{Wig}}(x)\,\mathrm{d}x$ are given by the kth Catalan number for each $k = 0, 1, 2, \ldots$. The functional form (1.1) now follows as a consequence of the Wigner semi-circle law being the unique probability density with such moments.

An extension of the question seeking the global density of a single Wigner matrix is to ask for the global density of the sum of matrices

$$Y = X^{(0)} + X, \tag{1.3}$$

where X is again a Wigner random matrix, but $X^{(0)}$ may be non-random, or random independent of X. To answer this a tool kit beyond the analysis of moments is required. Specifically, the analytic properties of the Stieltjes transform plays an essential role. Thus let $X^{(0)}/\sqrt{2N}$ have global spectral density $\rho^{(0)}(x)$, and define the corresponding Green's function (or Stieltjes transform) $G^{(0)}(z)$ by

$$G^{(0)}(z) = \int_{I^{(0)}} \frac{\rho^{(0)}(x)}{z-x}\,\mathrm{d}x, \tag{1.4}$$

where $I^{(0)}$ is the support of $\rho^{(0)}$. Then, with $G(z)$ denoting the Green's function of the sought global density $\rho(y)$ of $Y/\sqrt{2N}$, it is a known result (see e.g. [2, theorem 18.3.2]) that $G(z)$ is determined as the solution of the functional equation

$$G(z) = G^{(0)}\left(z - \frac{1}{4}G(z)\right), \tag{1.5}$$

subject to the requirements that $G(z) \sim 1/z$ as $|z| \to \infty$ and that $G(z)$ be analytic for $z \notin I$, where I is the support of $\rho(y)$.

As an illustration of (1.5), suppose

$$\rho^{(0)}(x) = \delta(x), \tag{1.6}$$

the Dirac delta function at the origin, in which case $G^{(0)}(z) = 1/z$. Then the functional equation (1.5) reads

$$G(z) = \frac{1}{z - G(z)/4},$$

and this has the solution

$$G(z) = 2\Big(z - \sqrt{z^2 - 1}\Big). \tag{1.7}$$

But according to the Sokhotski–Plemelj formula for the inverse of the Stieltjes transform,

$$\rho(y) = \frac{1}{2\pi \mathrm{i}} \lim_{\epsilon \to 0^+} (G(y - \mathrm{i}\epsilon) - G(y + \mathrm{i}\epsilon)), \qquad y \in I, \tag{1.8}$$

and substituting (1.7) reclaims (1.1).

Our interest is to develop a viewpoint of the functional equation (1.5) for the Green's function of the density for the matrix structure (1.3) as having origins in the hydrodynamical equation [9]

$$\frac{\partial \rho(x; \hat{\tau})}{\partial \hat{\tau}} = \frac{\partial}{\partial x}\Big(\rho(x; \hat{\tau}) \frac{\partial}{\partial x}\Big(V_1(x) + \int_{-\infty}^{\infty} \rho(y; \hat{\tau}) V_2(x, y)\, \mathrm{d}y \Big)\Big), \tag{1.9}$$

where $V_1(x), V_2(x, y)$ are particular one and two body potentials, and $\hat{\tau}$ is a scaled parameter. The scaling is required to compensate for the normalization of $\rho(y; \tau)$ being such that its total integral is unity rather than N. As to be reviewed in section 2, in the case that X in (1.3) is a standard Gaussian matrix, it is well known that a hydrodynamical equation, in particular the complex Burger's equation, relates to (1.5).

The advantage of viewing (1.5) as a consequence of (1.9) is that a hydrodynamical description applies equally well to the case that X and $X^{(0)}$ have the chiral structure

$$X = \begin{bmatrix} 0_{m\times m} & Z_{m\times n} \\ Z^{\dagger}_{n\times m} & 0_{n\times n} \end{bmatrix}, \qquad n \geqslant m, \tag{1.10}$$

and similarly $X^{(0)}$ with Z replaced by $Z^{(0)}$, where Z is a standard real or complex Gaussian matrix, and with $Z^{(0)}$ non-random or random independent of Z. The class of random matrices (1.10) is fundamental to random matrix theory (see e.g. [3, section 3.1]). Although a hydrodynamical description of the spectral evolution of complex chiral Gaussian matrices has been given in the recent works [4], the general solution of the resulting partial differential equation for the Green's function seems to have not been considered. In section 3 we show how (1.9) leads to a partial differential equation, transformable to an inhomogeneous complex Burger's equation. The partial differential equation was first derived in [5, 6], with the starting point in the latter being a microscopic description involving Dyson Brownian motion as reviewed in the introduction to section 3. Moreover, we show how to solve this partial differential equation in a form analogous to (1.5). In so doing we are able to reclaim a functional equation for the Green's function of so-called non-central Wishart matrices given by Dozier and Silverstein [7].

Specifically, consider random matrices of the form

$$W = \sqrt{\frac{2\hat{\tau}}{m}} \tilde{Z} + Z^{(0)},$$

where $\tilde{Z}, Z^{(0)}$ are rectangular $m \times n$ matrices, $\tilde{Z}$ a standard Gaussian and $Z^{(0)}$ fixed or random independent of $\tilde{Z}$. Let $\hat{a} = \lim_{n,m\to\infty}(n/m) - 1$, and let the $m, n \to \infty$ limiting density of eigenvalues of $Z^{(0)\dagger} Z^{(0)}$ be equal to $\rho^{W,(0)}(y)$, and the limiting density of eigenvalues of $W^{\dagger} W$ be equal to $\rho^{\mathrm{W}}(y; \hat{\tau})$. In this setting, and with

$$g^{\mathrm{W}}(z;\hat{\tau}) = \int_{I_+} \frac{\rho^{\mathrm{W}}(y;\hat{\tau})}{z-y}\,\mathrm{d}y, \tag{1.11}$$

$I_+ \subset R^+$ denoting the support of $\rho^{\mathrm{W}}(y;\hat{\tau})$, we show that

$$g^{\mathrm{W}}(z;\hat{\tau}) = F_W \int_{I_+^{(0)}} \frac{\rho^{W,(0)}(y)}{z(F_{\mathrm{W}})^2 - 2\hat{a}\hat{\tau}F_{\mathrm{W}} - y}\,\mathrm{d}y, \tag{1.12}$$

where $F_{\mathrm{W}} = 1 - 2\hat{\tau}g^{\mathrm{W}}(z;\hat{\tau})$. With the identifications $\hat{\tau} = \sigma^2 c/2$, $\hat{a} = (1-c)/c$ and $g^{\mathrm{W}} = -m$ this is equation (1.1) in [7].

When both X and $X^{(0)}$ in (1.3) are unitary, we are dealing with circular ensembles for which a hydrodynamical equation (1.9) is well known [8]. We review the necessary working in section 4 and proceed to present a novel hydrodynamical description of Jacobi ensemble in trigonometric variables, which can be interpreted as a 'chiral' deformation of circular ensembles. In sections 5 and 6, we turn to the collective variables approach which is used to review the well-known asymptotic expansion of Harish-Chandra/Itzykson–Zuber integral and to study not previously considered expansion of Berezin–Karpelevich type integrals.

2. Gaussian ensembles

In this section we review how the hydrodynamical equation (1.9) comes about from the study of the eigenvalues of the matrix sum (1.3) in the case that X is a standard Gaussian matrix with real ($\beta = 1$) or complex ($\beta = 2$) entries [9], and furthermore leads to the functional equation (1.5). The starting point is the fact that the Gaussian distribution on the space of matrices $\{Y\}$ which are real symmetric ($\beta = 1$) or complex Hermitian ($\beta = 2$),

$$P_\tau(X^{(0)};Y) = \frac{1}{C_{N,\tau}} \exp\left(-\frac{\beta}{4\tau}\,\mathrm{Tr}(Y - X^{(0)})^2\right), \tag{2.1}$$

where $C_{N,\tau}$ denotes the normalization constant, satisfies the diffusion equation

$$\frac{\partial P_\tau}{\partial \tau} = \frac{1}{\beta}\sum_\mu D_\mu \frac{\partial^2 P_\tau}{\partial Y_\mu^2}. \tag{2.2}$$

In (2.2) the label μ ranges over the independent elements, including both the real and imaginary parts of the off-diagonal elements if they are complex, and $D_\mu = 1$ for the diagonal elements, and $D_\mu = \frac{1}{2}$ for the off-diagonal elements.

There is also a class of Hermitian matrices with distribution (2.1) that satisfy (2.2) with $\beta = 4$. Thus the Hermitian matrix Y, and similarly $X^{(0)}$, is now a $2N \times 2N$ matrix formed from an $N \times N$ matrix with each element a 2×2 matrix of the form

$$\begin{bmatrix} z & -w \\ -\bar{w} & z \end{bmatrix}, \qquad z, w \in \mathbb{C}.$$

Such 2×2 matrices are isomorphic to the real quaternion division algebra, one of only three associative real normed division algebras along with the real and complex numbers, so in this case Y is said to have real quaternion entries; see e.g. [3, section 1.3.2].

The differential operator on the rhs of (2.2) can be interpreted as the Laplace–Beltrami operator ∇^2 associated with the metric form for the matrix spaces

$$(\mathrm{d}s)^2 = \mathrm{Tr}\left(\mathrm{d}Y \mathrm{d}Y^\dagger\right) = \sum_{\mu,\nu} g_{\mu\,\nu}\mathrm{d}Y_\mu \mathrm{d}Y_\nu,$$

with μ labeling the independent elements as in (2.2) and similarly ν, and where $g_{\mu\,\nu} = \dfrac{1}{D_\mu}\delta_{\mu,\nu}$; see e.g. [3, equation (11.9)]. Introducing the diagonalization formula for Y, $Y = ULU^\dagger$, where U is the matrix of eigenvectors and L is the diagonal matrix of the eigenvalues, allows ∇^2 to be rewritten according to the separated form

$$\nabla^2 = \frac{1}{J}\sum_{j=1}^{N}\frac{\partial}{\partial\lambda_j}\left(J\frac{\partial}{\partial\lambda_j}\right) + O_U, \qquad J = \prod_{j<k}\left|\lambda_k - \lambda_j\right|^\beta, \tag{2.3}$$

where the operator O_U involves derivatives with respect to variables relating to the eigenvectors only. The significance of this is that the eigenvalue distribution $p_\tau(\lambda_1,...,\lambda_N)$ obtained by integrating over the angles U and the distribution $P_0(X^{(0)})$ of $X^{(0)}$ in (2.1),

$$p_\tau\left(\lambda_1,...,\lambda_N\right) = J\int \mathrm{d}U \int \mathrm{d}X^{(0)} P_\tau\left(X^{(0)};\, ULU^\dagger\right)P_0\left(X^{(0)}\right) \tag{2.4}$$

satisfies the Smoluchowski–Fokker–Planck equation

$$\frac{\partial p_\tau}{\partial\tau} = \mathcal{L}p_\tau, \qquad \mathcal{L} = \sum_{j=1}^{N}\frac{\partial}{\partial\lambda_j}\left(\frac{\partial W}{\partial\lambda_j} + \beta^{-1}\frac{\partial}{\partial\lambda_j}\right), \tag{2.5}$$

with

$$W = -\sum_{1\leqslant j<k\leqslant N}\log|\lambda_j - \lambda_k|. \tag{2.6}$$

In the case that the τ dependence in (2.1) is modified so that P_τ satisfies not (2.2), but the heat equation for Brownian motion in a harmonic potential i.e. the Smoluchowski–Fokker–Planck equation (2.5) with W correspondingly modified by the addition of an harmonic potential $\frac{1}{2}\sum_{j=1}^{N}\lambda_j^2$, was first derived by Dyson [10]. As such the corresponding process is referred to as Dyson Brownian motion.

As pointed out in [10], the Smoluchowski–Fokker–Planck equation (2.5) has a more standard interpretation than its origin in random matrix theory. Specifically, consider a classical system of N particles interacting on a line with potential W. Suppose the particles execute overdamped Brownian motion in a fictitious background fluid with friction coefficient γ, and furthermore the system is at inverse temperature β. It is a basic fact—see e.g. [11]—that in this setting the evolution of the probability density $p_\tau(\lambda_1,...,\lambda_N)$ for the location of the particles at positions $\lambda_1,...,\lambda_N$ is given by the Smoluchowski–Fokker–Planck equation (2.5), where the lhs is to be multiplied by the friction coefficient γ. The random matrix problem gives rise to the specific potential (2.6), corresponding to the particles interacting via the repulsive pair potential $V_2(x, y) = -\log|x - y|$, and so the underlying classical gas is referred to as a log-gas [3].

Our interest is in the one-body dynamical density $\rho(x;\tau)$ defined as an average over (2.1),

$$\rho(x;\tau) = \frac{1}{N}\left\langle\sum_{i=1}^{N}\delta\left(x - \lambda_i\right)\right\rangle_{P_\tau}, \tag{2.7}$$

which has been normalized to have total integral unity. We probe a global regime for which the eigenvalues are scaled so that they have finite support as in (1.1). Note that since the integral over x of $\rho(x;\tau)$ is unity, the integral of $N\rho(x;\tau)$ over x is N. In this scaling the inter-particle spacing goes to zero, and the response of the system to perturbation is governed by macroscopic equations [9]. Relevant to the one-body dynamical density is the macroscopic

equation

$$N^2 J(x;\tau) = \mathcal{F}(x;\tau), \tag{2.8}$$

where $N^2 J(x;\tau)$ is the one-body current related to the density by the continuity equation

$$\frac{\partial}{\partial \hat{\tau}}\rho(x;\hat{\tau}) = -\frac{\partial}{\partial x}J(x;\hat{\tau}), \qquad \hat{\tau} = N\tau \tag{2.9}$$

(the scaled parameter $\hat{\tau}$ is introduced to compensate for the integral of $\rho(x;\tau)$ normalized to unity) while $\mathcal{F}(x;\tau)$ refers to the macroscopic force density. For the log-gas in the long wavelength regime the force density to leading order is of an electrostatics origin, so implying (see e.g. [12])

$$J(x;\tau) = -\rho(x;\tau)\frac{\partial}{\partial x}\Big(-\int_{-\infty}^{\infty}\rho(x';\tau)\log\Big|x-x'\Big|\,\mathrm{d}x'\Big). \tag{2.10}$$

Differentiating both sides with respect to x, and making use of the continuity equation (2.9) on the rhs, we see the hydrodynamical equation (1.9) results with $V_1(x) = 0$ and $V_2(x, y) = -\log|x-y|$.

We now want to show how this particular hydrodynamical equation leads to the functional equation (1.5). For this purpose, it is convenient to introduce the Hilbert (or Cauchy) transform as the principal value integral

$$\mathcal{H}[v](x) := \mathrm{PV}\int_I \frac{v(y)}{x-y}\,\mathrm{d}y, \qquad x \in I. \tag{2.11}$$

The hydrodynamical equation of interest then reads

$$\frac{\partial\rho(x;\hat{\tau})}{\partial\hat{\tau}} = -\frac{\partial}{\partial x}\Big(\rho(x;\hat{\tau})\mathcal{H}\big[\rho(\,\cdot\,;\hat{\tau})\big](x)\Big). \tag{2.12}$$

We also introduce the Green's function

$$G(z;\hat{\tau}) := \int_I \frac{\rho(y;\hat{\tau})}{z-y}\,\mathrm{d}y, \tag{2.13}$$

see (1.4).

Next we follow the working in [13, sections III.B.4], which begins by noting that as a consequence of the residue theorem, the Green's function is related to the Hilbert transform by

$$G_{\pm}(x;\hat{\tau}) = \mp \mathrm{i}\pi\rho(x;\hat{\tau}) + \mathcal{H}\big[\rho(\,\cdot\,;\hat{\tau})\big](x), \qquad x \in I. \tag{2.14}$$

Using this in (2.12) gives

$$2\frac{\partial}{\partial\hat{\tau}}\big(G_-(x;\hat{\tau}) - G_+(x;\hat{\tau})\big) = -\frac{\partial}{\partial x}\Big(\big(G_-(x;\hat{\tau})\big)^2 - \big(G_+(x;\hat{\tau})\big)^2\Big). \tag{2.15}$$

It must therefore be that the function

$$2\frac{\partial}{\partial\hat{\tau}}G(z;\hat{\tau}) + \frac{\partial}{\partial z}\big(G(z;\hat{\tau})\big)^2$$

is analytic throughout the entire complex plane. But according to (2.13), $G(z;\hat{\tau}) \sim 1/z$ as $|z| \to \infty$, so this function furthermore goes to zero at infinity. The only analytic function with this property is the zero function, and so after minor manipulation we have

$$\frac{\partial}{\partial\hat{\tau}}G\big(z;\hat{\tau}\big)+G\big(z;\hat{\tau}\big)\frac{\partial}{\partial z}G\big(z;\hat{\tau}\big)=0. \tag{2.16}$$

This is the Euler equation, or equivalently complex Burgers equation of hydrodynamics. Thus with $z=x+\mathrm{i}y$ and $G(z;\tau)=U+\mathrm{i}V$, (U,V) is the velocity field at point (x,y) in the plane for an ideal fluid at constant pressure.

To solve the initial value problem for (2.16), we invoke the method of complex characteristics[14, 15] where both z and G are complex functions. This is a slight generalization of a standard technique applicable to initial value problems of real first order PDEs. We present this method for a general first order equation of the form

$$A\big(G\big(z;\hat{\tau}\big),z,\hat{\tau}\big)\frac{\partial}{\partial\hat{\tau}}G\big(z;\hat{\tau}\big)+B\big(G\big(z;\hat{\tau}\big),z,\hat{\tau}\big)\frac{\partial}{\partial z}G\big(z;\hat{\tau}\big)=C\big(G\big(z;\hat{\tau}\big),z,\hat{\tau}\big). \tag{2.17}$$

The main idea is to seek a coordinate transform $(z,\hat{\tau})\to(\alpha,\beta)$ such that the PDE (2.17) becomes an ODE along the curves of constant α,

$$\frac{\mathrm{d}}{\mathrm{d}\beta}G\big(z;\hat{\tau}\big)=C\big(G\big(z;\hat{\tau}\big),z,\hat{\tau}\big) \tag{2.18}$$

called characteristic lines or simply characteristics. By the chain rule $\frac{\mathrm{d}}{\mathrm{d}\beta}=\frac{\mathrm{d}\hat{\tau}}{\mathrm{d}\beta}\frac{\partial}{\partial\hat{\tau}}+\frac{\mathrm{d}z}{\mathrm{d}\beta}\frac{\partial}{\partial z}$, the left-hand sides of (2.18) and (2.17) dictate the system of equations describing the characteristics,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\beta}\hat{\tau}(\alpha,\beta)&=A\big(G(\alpha,\beta),z(\alpha,\beta),\hat{\tau}(\alpha,\beta)\big),\\ \frac{\mathrm{d}}{\mathrm{d}\beta}z(\alpha,\beta)&=B\big(G(\alpha,\beta),z(\alpha,\beta),\hat{\tau}(\alpha,\beta)\big),\end{aligned} \tag{2.19}$$

where $G(\alpha,\beta)=G(z(\alpha,\beta);\hat{\tau}(\alpha,\beta))$. These equations form lines in the $(z,\hat{\tau})$ space, labeled by the β parameter and passing through the prescribed initial point $(z(\alpha,0),\hat{\tau}(\alpha,0))$. For the latter to be determined, the Green's function $G(z;\hat{\tau}=0)$ on the $\hat{\tau}=0$ line is required. With this initial data specified, the set of equations (2.18) and (2.19) are in principle solvable by standard means and comprise the sought solution to (2.17).

The equation (2.16) is an instance of (2.17) with $A=1$, $B=G$ and $C=0$. We read off from (2.17) and (2.19) that the differential equations describing characteristic lines and the propagation of the solution are

$$\frac{\mathrm{d}}{\mathrm{d}\beta}z(\alpha,\beta)=G(\alpha,\beta),\qquad\frac{\mathrm{d}}{\mathrm{d}\beta}\hat{\tau}(\alpha,\beta)=1,\qquad\frac{\mathrm{d}}{\mathrm{d}\beta}G(\alpha,\beta)=0. \tag{2.20}$$

The initial data comprises of the initial position $z(\alpha,0)=\alpha$, $\hat{\tau}(\alpha,0)=0$ and the starting Green's function $G(\alpha,0)=G(z(\alpha,0);\hat{\tau}(\alpha,0))=G(\alpha;0)$. Explicit integration gives

$$\begin{aligned}G(\alpha,\beta)&=G(\alpha;0),\\ \hat{\tau}(\alpha,\beta)&=\beta,\\ z(\alpha,\beta)&=\alpha+\beta G(\alpha;0).\end{aligned} \tag{2.21}$$

These, after eliminating α and β, yield the functional equation

$$G\big(z;\hat{\tau}\big)=G\big(z-\hat{\tau}G\big(z;\hat{\tau}\big);0\big), \tag{2.22}$$

which is to be compared to (1.5). Equivalently, recalling the definition (1.4), we obtain the implicit integral equation

$$G\big(z;\hat{\tau}\big)=\int_{I^{(0)}}\frac{\rho^{(0)}(\mu)\mathrm{d}\mu}{z-\hat{\tau}G\big(z;\hat{\tau}\big)-\mu}. \tag{2.23}$$

Working closely related to the above discussion can be found in [16].

To anticipate the precise relationship between (2.22) and (1.5), let us choose $\tau=1/(4N)$. The rhs of (2.1) is then proportional to $\exp\left(-N\beta(Y-X^{(0)})^2\right)$, and thus we see that $Y=X+X^{(0)}$, where $X=\tilde{X}/\sqrt{2N}$ with $\tilde{X}$ a standard Gaussian matrix. This is precisely the setting which gives rise to (1.5). On the other hand, the choice $\tau=1/(4N)$ is, according to (2.9), equivalent to the choice $\hat{\tau}=1/4$, and this substituted in (2.22) gives (1.5).

In the Introduction the functional equation (1.5) was illustrated by showing that the case $\rho^{(0)}(x)=\delta(x)$ leads to the Wigner semi-circle law (1.1). Another example which permits an explicit functional form for the density is when

$$G^{(0)}(z)=\frac{1}{2}\left(\frac{1}{z-a}+\frac{1}{z+a}\right). \tag{2.24}$$

This corresponds to an initial density

$$\rho^{(0)}(x)=\frac{1}{2}(\delta(x-a)+\delta(x+a)), \tag{2.25}$$

or equivalently to $X^{(0)}$ in (1.3) being a diagonal matrix with half its eigenvalues at a and the other half at $-a$. Substituting (2.24) in (2.22) with $\hat{\tau}=1$ shows that the Green's function $G(z)$ satisfies the cubic equation

$$G^3-2zG^2+\big(1-a^2+z^2\big)G-z\;=\;0. \tag{2.26}$$

With $\xi=z-G$, this equation first appeared in the present context in [17], where it was shown to correspond to a spectral density supported on two disjoint intervals symmetrical about the origin for $a>1$. In the case $a=1$ the intervals meet at the origin, and it can then be shown that the eigenvalue density has the explicit form [18, equation (6.118)]

$$\rho(y)=\frac{|y|^{1/3}}{2\sqrt{3}\,\pi}\left(\left(3\sqrt{3}+\sqrt{27-8y^2}\right)^{2/3}-\left(3\sqrt{3}-\sqrt{27-8y^2}\right)^{2/3}\right) \tag{2.27}$$

for $-\sqrt{27/8}\leqslant y\leqslant\sqrt{27/8}$. For a discussion of (2.26) in terms of caustics corresponding to the complex Burger's equation, see [16].

3. Chiral Gaussian ensembles

3.1. Partial differential equation

We now turn our attention to the matrix sum (1.3) in the case that X is a chiral Gaussian random matrix as specified by (1.10), and $X^{(0)}$ has the same block structure as X but is non-random or random independent of X. Thus the matrix Y in (1.3) similarly has the block structure

$$Y=\begin{bmatrix}0_{m\times m} & W_{m\times n}\\ W^{\dagger}_{n\times m} & s0_{n\times n}\end{bmatrix},\qquad W\;=\;Z+Z_0. \tag{3.1}$$

Instead of (2.1), we now have a distribution on the block matrix W specified by

$$P_\tau\big(Z^{(0)};\, W\big) = \frac{1}{C^{\mathrm{c}}_{N,\tau}} \exp\left(\frac{-\beta}{4\tau} \mathrm{Tr}\big(W - Z^{(0)}\big)^\dagger \big(W - Z^{(0)}\big)\right). \tag{3.2}$$

The diffusion equation (2.1), but with Y replaced by W, again allows for a characterization of this matrix distribution. Before the (Hermitian) matrix Y was decomposed according to the diagonalization formula. The appropriate decomposition of the matrix W is the singular value decomposition $W = ULV^\dagger$, where U and V are real orthogonal ($\beta = 1$) or complex unitary ($\beta = 2$) matrices of size $m \times m$ and $n \times n$ respectively, while $L = \mathrm{diag}(x_1, ..., x_m)$, where $\{x_j\}$ are the singular values of W or equivalently $\{x_j^2\}$ are the eigenvalues of $W^\dagger W$. As discussed in [19, 20], integrating over the distribution of $Z^{(0)}$ leaves a distribution function $p_\tau(x_1^2, ..., x_m^2)$ depending on the parameter τ and the eigenvalues $W^\dagger W$ only. To characterize this distribution as the solution of an evolution equation, we require the fact (see e.g. [3, section 11.2.2]) that the Jacobian J in the formula (2.3) should now read

$$J = \prod_{j=1}^m x_j^{\beta a + 1} \prod_{1 \leqslant j < k \leqslant m} |x_k^2 - x_j^2\,|^\beta, \qquad a = n - m + 1 - 2/\beta,$$

and furthermore on the rhs the replacements

$$\big\{\lambda_j\big\} \mapsto \{x_j\}, \qquad N \mapsto m \tag{3.3}$$

should be made. Doing this then gives that $p_\tau(x_1^2, ..., x_m^2)$ satisfies the Smoluchowski–Fokker–Planck equation (2.5) with

$$W = -\frac{a'}{2} \sum_{j=1}^m \log x_j^2 - \sum_{1 \leqslant j < k \leqslant m} \log |x_k^2 - x_j^2|, \tag{3.4}$$

where $a' = a + 1/\beta$ and with the replacements (3.3). In the log-gas analogy, the domain is now the half line $x > 0$, and there is both a one and two body potential given by

$$V_1(x) = -\frac{a'}{2} \log x^2, \qquad V_2(x, y) = -\log \big|x^2 - y^2\big|. \tag{3.5}$$

We now turn our attention to the hydrodynamical description of the global density $\rho^{\mathrm{c}}(x; \hat{\tau})$ which is defined as

$$\rho^{\mathrm{c}}\big(x; \hat{\tau}\big) = \frac{1}{m} \left\langle \sum_{i=1}^m \big(\delta(x - x_i) + \delta(x + x_i)\big) \right\rangle_{P_\tau}, \tag{3.6}$$

where we average over the measure (3.2), and the superscript 'c' denotes the chiral case. This density is an even function in x and is normalized so that integration over the positive half line $x > 0$ gives unity. As in the discussion of section 2, to access the global regime we must scale the parameter $\hat{\tau} = m\tau$, and also scale

$$\hat{a} = \lim_{\substack{n \to \infty \\ m \to \infty}} \frac{n}{m} - 1, \tag{3.7}$$

which so determines the limiting ratio n/m. In terms of these scaled parameters, from the explicit form (3.5) of the one and two body potentials, and the fact that the domain is a half line, the hydrodynamical equation (1.9) reads

$$\frac{\partial \rho^{\mathrm{c}}(x;\hat{\tau})}{\partial \hat{\tau}} = \frac{\partial}{\partial x}\left(\rho^{\mathrm{c}}(x;\hat{\tau})\frac{\partial}{\partial x}\left(-\frac{\hat{a}}{2}\log x^2 - \int_0^\infty \rho^{\mathrm{c}}(y;\hat{\tau})\log\left|x^2-y^2\right|\,\mathrm{d}y\right)\right)$$
$$= \frac{\partial}{\partial x}\left(\rho^{\mathrm{c}}(x;\hat{\tau})\frac{\partial}{\partial x}\left(-\frac{\hat{a}}{2}\log x^2 - \int_{-\infty}^\infty \rho^{\mathrm{c}}(y;\hat{\tau})\log|x-y|\,\mathrm{d}y\right)\right), \tag{3.8}$$

where the second line follows by writing $\log|x^2-y^2| = \log|x-y| + \log|x+y|$ and the fact that $\rho^{\mathrm{c}}(y;\hat{\tau})$ is even in y.

Introducing the Hilbert transform as defined in (2.11), (3.8) can be written

$$\frac{\partial \rho^{\mathrm{c}}(x;\hat{\tau})}{\partial \hat{\tau}} = -\frac{\partial}{\partial x}\Big(\rho^{\mathrm{c}}(x;\hat{\tau})\mathcal{H}\big[\hat{a}\delta(\,\cdot\,) + \rho^{\mathrm{c}}(\,\cdot\,;\hat{\tau})\big](x)\Big). \tag{3.9}$$

Introducing too the Green's function

$$G^{\mathrm{c}}(z;\hat{\tau}) = \int_{I_+\bigcup -I_+} \frac{\rho^{\mathrm{c}}(y;\hat{\tau})}{z-y}\,\mathrm{d}y, \tag{3.10}$$

where $I_+ \subset \mathbb{R}^+$ is the support of $\rho^{\mathrm{c}}(y;\hat{\tau})$ on the positive real axis, we see that

$$G^{\mathrm{c}}_{\pm}(x;\hat{\tau}) = \mp \mathrm{i}\pi\rho^{\mathrm{c}}(x;\hat{\tau}) + \mathcal{H}\big[\rho^{\mathrm{c}}(\,\cdot\,;\hat{\tau})\big](x), \qquad x \in I_+ \bigcup -I_+, \tag{3.11}$$

where $G^{\mathrm{c}}_{\pm}$ is defined according to (2.14). Proceeding as in the derivation of (2.16), it follows from the use of (3.11) in (3.9) that

$$\frac{\partial}{\partial\hat{\tau}}G^{\mathrm{c}}(z;\hat{\tau}) - \frac{\hat{a}}{z^2}G^{\mathrm{c}}(z;\hat{\tau}) + \left(G^{\mathrm{c}}(z;\hat{\tau}) + \frac{\hat{a}}{z}\right)\frac{\partial}{\partial z}G^{\mathrm{c}}(z;\hat{\tau}) = 0. \tag{3.12}$$

3.2. Solution of the partial differential equation

We now seek the general solution of the initial value problem for this partial differential equation, first in the case $\hat{a}=0$, then in the more difficult case $\hat{a}>0$.

The case $\hat{a}=0$.. In the case $\hat{a}=0$ we see that (3.12) reduces to the Euler equation (2.16). Thus as with (2.22), the solution in this case is

$$G^{\mathrm{c}}(z;\hat{\tau}) = G^{\mathrm{c}}\big(z - \hat{\tau}G^{\mathrm{c}}(z;\hat{\tau});\,0\big), \tag{3.13}$$

but we must keep in mind that G^{c} is defined by (3.10) rather than (2.13). For (2.22) and (3.13) to imply identical, up to a scale factor, eigenvalue distributions and singular value distributions respectively for general $\hat{\tau}$, we see that we must have the initial conditions related by

$$\rho(y;0) = \frac{1}{2}\rho^{\mathrm{c}}(y;0), \qquad y \in I_+^{(0)} \bigcup -I_+^{(0)}, \tag{3.14}$$

where $I_+^{(0)}$ is the support of $\rho^{\mathrm{c}}(y;0)$ on the positive real axis. The factor of $\frac{1}{2}$ is to compensate for the normalization of the lhs being such that integration over the whole real line gives unity, while on the rhs integration of $\rho^{\mathrm{c}}(y;0)$ over the half line $y>0$ gives unity which is evident from the definition (3.6).

Specifically, we see that with the initial conditions related by (3.14), the solutions of (2.22) and (3.13) are related by

$$G(z;\hat{\tau}) = \frac{1}{2}G^{\mathrm{c}}\left(z;\frac{\hat{\tau}}{2}\right)$$

and thus

$$\rho(y;\hat{\tau}) = \frac{1}{2}\rho^{\mathrm{c}}\left(y;\frac{\hat{\tau}}{2}\right). \tag{3.15}$$

In relation to the initial condition $\rho(y;0) = \delta(y)$, after recalling that the Wigner semi-circle law (1.1) results from the parameter value $\hat{\tau} = 1/4$, we see from (3.15) that

$$\rho^{\mathrm{c}}(x;\hat{\tau}) = \frac{1}{2\pi\hat{\tau}}\sqrt{8\hat{\tau} - x^2}, \qquad 0 \leqslant x \leqslant 2\sqrt{2\hat{\tau}}. \tag{3.16}$$

It must therefore be that for $Z^{(0)} = 0_{m\times n}$ in (3.2) the global density of the singular values of the $m \times n$, $n > m$, matrix W with distribution specified by (3.2), is in the case that $\lim_{m\to\infty}\frac{n}{m} = 1$ equal to this functional form.

To see this latter result, which is well known, first note that each singular value x of W is related to an eigenvalue y of $W^\dagger W$ by $x^2 = y$. Changing variables according to this prescription in (3.16) gives the density function

$$\frac{1}{4\pi\hat{\tau}\sqrt{y}}\sqrt{8\hat{\tau} - y}, \qquad 0 < y < 8\hat{\tau}. \tag{3.17}$$

With $\hat{\tau} = \frac{1}{8}$ this specifies the Marchenko–Pastur law for the limiting density of the eigenvalues of the scaled matrices $\frac{1}{4m}W^\dagger W$, with W now a standard Gaussian rectangular matrix, again in the circumstance that $\lim_{m\to\infty}\frac{n}{m} = 1$; see e.g. [3, section 3.4.1]. This scaling is consistent with that implied by (3.1), with $Z^{(0)} = 0_{m\times n}$ and $\tau = \hat{\tau}/m = 1/(8m)$.

An analogous discussion holds for the initial density (2.24). In the case $a = 1$, $\hat{\tau} = 1/2$, and upon changing variables $y^2 = x$, we conclude that the limiting density of the eigenvalues of the scaled matrices $\frac{1}{m}(X + Z^{(0)})^\dagger(X + Z^{(0)})$, where X is a standard Gaussian $m \times n$ rectangular matrix $Z^{(0)}$ is an $m \times n$ matrix with half its entries on the diagonal equal to $+1$, the other half -1, all other entries equal to 0, is equal to

$$\frac{1}{2^{5/3}3^{1/2}\pi}\frac{\left(3\sqrt{3} + \sqrt{27 - 4x}\right)^{2/3} - \left(3\sqrt{3} - \sqrt{27 - 4x}\right)^{2/3}}{x^{1/3}}, \tag{3.18}$$

supported on $0 < x \leqslant \frac{27}{4}$. This is the density function for the Raney distribution with parameters $p = 3$, $r = 2$ [21], where for general $0 < r \leqslant p$ and $p > 1$ the Raney distribution is characterized by its kth moments according to

$$R_{p,r}(k) = \frac{r}{pk + r}\binom{pk + r}{k}, \qquad k = 0, 1, 2, \ldots. \tag{3.19}$$

Note that $R_{2,1}$ corresponds to the Catalan numbers; recall (1.2).

The case $\hat{a} > 0$. For nonzero parameter $\hat{a}$, the solution of (3.12) to an initial value problem is given by the method of characteristics described in section 2. The chiral ensemble is an instance of (2.17) with $A = 1$, $B = G^{\mathrm{c}} + \hat{a}/z$ and $C = \hat{a}G^{\mathrm{c}}/z^2$. Accordingly, the system of ODEs describing both the characteristic lines and the propagation of Green's function is

$$\frac{\mathrm{d}}{\mathrm{d}\beta}z(\alpha,\,\beta)=G^{\mathrm{c}}(\alpha,\,\beta)+\frac{\hat{a}}{z(\alpha,\,\beta)},$$
$$\frac{\mathrm{d}}{\mathrm{d}\beta}G^{\mathrm{c}}(\alpha,\,\beta)=\hat{a}\frac{G^{\mathrm{c}}(\alpha,\,\beta)}{z(\alpha,\,\beta)^2},$$
$$\frac{\mathrm{d}}{\mathrm{d}\beta}\hat{\tau}(\alpha,\,\beta)=1,$$

with initial conditions $z(\alpha,\,0)=\alpha$, $\hat{\tau}(\alpha,\,0)=0$ and

$$G^{\mathrm{c}}(\alpha,\,0)G^{\mathrm{c}}(z(\alpha,\,0);\,\hat{\tau}(\alpha,\,0))G^{\mathrm{c}}(\alpha;\,0).$$

The last equation for $\hat{\tau}$ is simply solved as $\hat{\tau}=\beta$ whereas the first two are coupled but readily solved to give

$$z(\alpha,\,\beta)=\sqrt{\alpha+G_0^{\mathrm{c}}(\alpha)\beta}\sqrt{\alpha+G_0^{\mathrm{c}}(\alpha)\beta+2\hat{a}\frac{\beta}{\alpha}},\qquad \hat{\tau}(\alpha,\,\beta)=\beta, \tag{3.20}$$

$$G^{\mathrm{c}}(\alpha,\,\beta)=G_0^{\mathrm{c}}(\alpha)\frac{\sqrt{\alpha+G_0^{\mathrm{c}}(\alpha)\beta+2\hat{a}\frac{\beta}{\alpha}}}{\sqrt{\alpha+G_0^{\mathrm{c}}(\alpha)\beta}}, \tag{3.21}$$

where we used a simplified notation $G_0^{\mathrm{c}}(\alpha)=G^{\mathrm{c}}(\alpha,\,0)$. Now we make the substitution $\hat{\tau}=\beta$ and calculate auxiliary formulas by multiplying the equations for z and G^{c} and squaring the equation for z,

$$\frac{G^{\mathrm{c}}\big(z;\,\hat{\tau}\big)}{z}=\frac{G_0^{\mathrm{c}}(\alpha)}{\alpha+G_0^{\mathrm{c}}(\alpha)\hat{\tau}}, \tag{3.22}$$

$$z^2\alpha=\big(\alpha+G_0^{\mathrm{c}}(\alpha)\hat{\tau}\big)^2\alpha+2\hat{a}\hat{\tau}\big(\alpha+G_0^{\mathrm{c}}(\alpha)\hat{\tau}\big). \tag{3.23}$$

From (3.22) we find $G_0^{\mathrm{c}}=\dfrac{\alpha G^{\mathrm{c}}}{z-G^{\mathrm{c}}\hat{\tau}}$ and from that

$$\alpha+G_0^{\mathrm{c}}\hat{\tau}=\frac{\alpha z}{z-G^{\mathrm{c}}\hat{\tau}}, \tag{3.24}$$

where from now on we suppress the arguments of G^{c} and G_0^{c} for brevity. We plug (3.24) into (3.23) to obtain a formula for α^2,

$$\alpha^2=\big(z-G^{\mathrm{c}}\hat{\tau}\big)^2-\frac{2\hat{a}\hat{\tau}\big(z-G^{\mathrm{c}}\hat{\tau}\big)}{z}. \tag{3.25}$$

Next we recall the definition (3.10) to determine G_0^{c} and use the symmetry of the initial spectral density $\rho^{c,(0)}$ to rewrite

$$G_0^{\mathrm{c}}(\alpha)=\int_{I_+^{(0)}\cup -I_+^{(0)}}\frac{\rho^{c,(0)}(\mu)\mathrm{d}\mu}{\alpha-\mu}=2\alpha\int_{I_+^{(0)}}\frac{\rho^{c,(0)}(\mu)\mathrm{d}\mu}{\alpha^2-\mu^2}. \tag{3.26}$$

This variant of G_0^{c} plugged into (3.22) gives

$$G^{\mathrm{c}}=\frac{z-G^{\mathrm{c}}\hat{\tau}}{\alpha}G_0^{\mathrm{c}}(\alpha)=2\big(z-G^{\mathrm{c}}\hat{\tau}\big)\int_{I_+^{(0)}}\frac{\rho^{c,(0)}(\mu)\mathrm{d}\mu}{\alpha^2-\mu^2}, \tag{3.27}$$

where we also used (3.24). Lastly, we recall the equation (3.25) for α^2 and so obtain an implicit integral equation

$$G^{\mathrm{c}}(z;\hat{\tau}) = 2\int_{I_+^{(0)}} \frac{\rho^{c,(0)}(\mu)\mathrm{d}\mu}{z - G^{\mathrm{c}}(z;\hat{\tau})\hat{\tau} - \dfrac{2\hat{a}\hat{\tau}}{z} - \dfrac{\mu^2}{z - G^{c}(z;\hat{\tau})\hat{\tau}}}. \tag{3.28}$$

We can check, upon recalling (3.10), that in the case $\hat{a} = 0$ (3.28) is equivalent to (3.13).

As a first illustration of (3.28), let $\rho^{c,(0)}(\mu) = 2\delta(\mu)$, Then (3.28) simplifies to a quadratic for G^{c}, and from this use of the Sokhotski–Plemelj formula (1.8) implies the density of singular values is equal to

$$\frac{1}{\pi x}\big(-x^4 + (2\hat{a} + 4)x^2 - \hat{a}^2\big)^{1/2}, \tag{3.29}$$

where we have chosen $\hat{\tau} = 1/2$ as in deriving (3.18), supported on the region of the positive real axis such that the argument of the square root is positive. This result is well known; see e.g. [3, proposition 3.4.1].

As a second illustration, suppose $\rho^{c,(0)}(\mu) = \delta(\mu - b) + \delta(\mu + b)$ so that the singular values of $Z^{(0)}$ are all located at b. We then find that (3.28) gives a cubic equation for G^{c},

$$g\big(z^2(1-g)^2 - \hat{a}(1-g) - b^2\big) = 1 - g, \tag{3.30}$$

where we have set $\hat{\tau} = 1/2$, and $g := G^{\mathrm{c}}\hat{\tau}/z$. For $\hat{a} > 0$, we can see from this that the density is supported away from the origin. The reasoning is that otherwise, for small z, g must behave like $z^{-1-\alpha}$ with $0 < \alpha < 1$, as would follow from its relation to G^{c} and (3.10). But this is incompatible with (3.30) unless $\hat{a} = 0$.

Equivalent viewpoints. The Green's function (3.10) is the Stieltjes transform of the density of the singular values of the matrix W in (3.1). The singular values of W also appear as the eigenvalues of the block matrix Y specified in (3.10). Thus Y has $n - m$ zero eigenvalues, m eigenvalues equal to the singular values of W, and m eigenvalues equal to minus the singular values of W; see e.g. [3, proposition 3.1.1]. Denoting the corresponding density, normalized to integrate to unitary, by $\rho^{\mathrm{ch}}(x;\tau)$ we see that

$$\rho^{\mathrm{ch}}(x;\tau) = \frac{1}{2+\hat{a}}\rho^{\mathrm{c}}(x;\tau) + \frac{\hat{a}}{2+\hat{a}}\delta(x). \tag{3.31}$$

Hence, the corresponding Green's function

$$g^{\mathrm{ch}}(z;\tau) := \int_{I_+\cup -I_+} \frac{\rho^{\mathrm{ch}}(x;\tau)}{z-x}\,\mathrm{d}x$$

is related to the Green's function (3.10) by

$$g^{\mathrm{ch}}(z;\hat{\tau}) = \frac{1}{2+\hat{a}}G^{\mathrm{c}}(z;\tau) + \frac{\hat{a}}{2+\hat{a}}\frac{1}{z}. \tag{3.32}$$

Substituting in (3.12) gives the inhomogeneous Burger's equation

$$\frac{\partial}{\partial\hat{\tau}}g^{\mathrm{ch}}(z;\hat{\tau}) + (2+\hat{a})g^{\mathrm{ch}}(z;\hat{\tau})\frac{\partial}{\partial z}g^{\mathrm{ch}}(z;\hat{\tau}) + \frac{\hat{a}^2}{(2+\hat{a})}\frac{1}{z^3} = 0, \tag{3.33}$$

which is the form considered in [22] for the present setting.

Another variant is to consider the eigenvalues of $W^\dagger W$, with W as in (3.1). As already remarked below (3.2), the eigenvalues are $W^\dagger W$ are the squared singular values of W. Thus, with the corresponding density denoted by $\rho^{\mathrm{W}}(X;\tau)$, we have

$$\rho^{\mathrm{W}}(X;\,\tau) = \frac{1}{2\sqrt{X}}\rho^{\mathrm{c}}\Big(\sqrt{X};\,\tau\Big). \tag{3.34}$$

The Green's function for the eigenvalues of W is given in terms of $\rho^{\mathrm{W}}(X;\,\tau)$ by

$$g^{\mathrm{W}}\big(z;\,\hat{\tau}\big) = \int_{I_+} \frac{\rho^{\mathrm{W}}\big(x;\,\hat{\tau}\big)}{z-x}\,\mathrm{d}x.$$

Recalling (3.10) we thus have the relation

$$g^{\mathrm{W}}\big(z^2;\,\hat{\tau}\big) = \frac{G^{\mathrm{c}}\big(z;\,\hat{\tau}\big)}{2z}. \tag{3.35}$$

Substituting in (3.12) gives that g^{W} satisfies the Burger's like equation

$$\frac{\hat{a}+1}{2}\frac{\partial}{\partial\hat{\tau}}g^{\mathrm{W}}\big(z;\,\hat{\tau}\big) + \big(\hat{a}+2zg^{\mathrm{W}}\big(z;\,\hat{\tau}\big)\big)\frac{\partial}{\partial z}g^{\mathrm{W}}\big(z;\,\hat{\tau}\big) + g^{\mathrm{W}}\big(z;\,\hat{\tau}\big)^2 \;=\; 0. \tag{3.36}$$

This variant, modulo some rescaling, is the one given in [5, 6] in the present context.

For completeness we present the solution to the initial value problem of both (3.33) and (3.36), which follow from (3.28) by substitution. In the case of g^{ch}, substituting (3.31) and (3.32) in (3.28) we find

$$g^{\mathrm{ch}}\big(z;\,\hat{\tau}\big) = 2F_{\mathrm{ch}}\int_{I_+^{(0)}} \frac{\rho^{\mathrm{ch},(0)}(\mu)\mathrm{d}\mu}{F_{\mathrm{ch}}^2 - \dfrac{F_{\mathrm{ch}}\hat{a}\hat{\tau}}{z} - \mu^2} + \frac{\hat{a}}{2+\hat{a}}\left(\frac{1}{z} - \frac{2z}{zF_{\mathrm{ch}} - \hat{a}\hat{\tau}}\right), \tag{3.37}$$

where $F_{\mathrm{ch}} = z - \hat{\tau}(2+\hat{a})g^{\mathrm{ch}}$. In the case of (3.36), substituting (3.35) in (3.28) gives (1.12).

With (1.12) of interest in mathematical statistics and thus as a stand alone result, let us show the method of characteristics discussed in section 2 can be used to solve the partial differential equation (3.36) directly. We first note (3.36) is an instance of (2.17) with replacements $G = g^{\mathrm{W}}$ and $A = \dfrac{1}{2}$, $B = \hat{a} + 2zg^{\mathrm{W}}$, $C = -(g^{\mathrm{W}})^2$. The ODEs (2.18) and (2.19) read

$$\frac{\partial}{\partial\beta}\hat{\tau}(\alpha,\,\beta) = \frac{1}{2}, \tag{3.38}$$

$$\frac{\partial}{\partial\beta}z(\alpha,\,\beta) = \hat{a} + 2z(\alpha,\,\beta)g^{\mathrm{W}}(\alpha,\,\beta), \tag{3.39}$$

$$\frac{\partial}{\partial\beta}g^{\mathrm{W}}(\alpha,\,\beta) = -g^{\mathrm{W}}(\alpha,\,\beta)^2, \tag{3.40}$$

where $(\alpha,\,\beta)$ are the transformed variables $(z,\,\hat{\tau})$ and the abbreviated notation $g^{\mathrm{W}}(\alpha,\,\beta) = g^{\mathrm{W}}(z(\alpha,\,\beta);\,\hat{\tau}(\alpha,\,\beta))$. The solution to these ODE's with initial conditions $z(\alpha,\,0) = \alpha, \hat{\tau}(\alpha,\,0) = 0$ and $g^{\mathrm{W}}(\alpha,\,0) = g_0^{\mathrm{W}}(\alpha)$ is

$$\hat{\tau}(\alpha,\,\beta) = \frac{\beta}{2}, \tag{3.41}$$

$$z(\alpha,\,\beta) = \hat{a}\beta\Big(1 + g_0^{\mathrm{W}}(\alpha)\beta\Big) + \alpha\Big(1 + g_0^{\mathrm{W}}(\alpha)\beta\Big)^2, \tag{3.42}$$

$$g^{\mathrm{W}}(\alpha,\,\beta) = \frac{g_0^{\mathrm{W}}(\alpha)}{1 \;+\; g_0^{\mathrm{W}}(\alpha)\beta}. \tag{3.43}$$

Since to proceed requires purely algebraic operations only, we suppress all of the arguments in what follows. We find a formula for $g_{W,0}$ from the last equation,

$$g_0^{\mathrm{W}} = \frac{g^{\mathrm{W}}}{1 - g^{\mathrm{W}}\beta}, \tag{3.44}$$

and substitute it into (3.42) to obtain the formula for α,

$$\alpha = z\big(1 - g^{\mathrm{W}}\beta\big)^2 - \hat{a}\beta\big(1 - g^{\mathrm{W}}\beta\big). \tag{3.45}$$

Next we turn to (3.44) and find

$$g^{\mathrm{W}} = \big(1 - g^{\mathrm{W}}\beta\big)g_0^{\mathrm{W}} \tag{3.46}$$

which, including the formula (3.45) for α and (3.41), gives an implicit solution of (3.36)

$$g^{\mathrm{W}}\big(z;\hat{\tau}\big) = \big(1 - 2\hat{\tau}g^{\mathrm{W}}\big(z;\hat{\tau}\big)\big)g_0^{\mathrm{W}}\Big(z\big(1 - 2\hat{\tau}g^{\mathrm{W}}\big(z;\hat{\tau}\big)\big)^2 - 2\hat{a}\hat{\tau}\big(1 - 2\hat{\tau}g^{\mathrm{W}}\big(z;\hat{\tau}\big)\big)\Big). \tag{3.47}$$

Since

$$g_0^{\mathrm{W}}(z) = \mathrm{PV}\int_{I_+^{(0)}} \frac{\rho^{W,(0)}(\mu)\mathrm{d}\mu}{z - \mu} \tag{3.48}$$

with $I_+^{(0)} \subset R^+$ and $\rho^{W,(0)}(\mu)$ is normalized to unity when integrating over $\mu > 0$ we see that (1.12) follows.

4. Circular and Jacobi ensembles

4.1. Circular ensembles

We now turn our attention to Smoluchowski–Fokker–Planck type dynamics of circular ensembles. These models arise when considering random (symmetric if $\beta = 1$, unrestricted if $\beta = 2$ or self-dual if $\beta = 4$) unitary matrices U of size $N \times N$ distributed according to the Haar measure; see e.g. [3, ch 2]. The Jacobian in this case reads

$$J = \prod_{1<i<j<N} |\mathrm{e}^{\mathrm{i}\phi_i} - \mathrm{e}^{\mathrm{i}\phi_j}|^\beta, \tag{4.1}$$

so that eigenvalues $\mathrm{e}^{\mathrm{i}\phi_i}$ lie on a unit circle. The diffusion is introduced based on the parametrization of U in terms of exponent of an Hermitian matrix. The joint eigenvalue PDF satisfies (2.5) with a drift term

$$W = -\sum_{1<i<j<N} \log|\mathrm{e}^{\mathrm{i}\phi_i} - \mathrm{e}^{\mathrm{i}\phi_j}| \tag{4.2}$$

and with the replacements $\{\lambda_i\} \mapsto \{\phi_i\}$.; see e.g. [3, section 11.2.1].

The log-gas system occupies the domain $\phi \in (-\pi, \pi]$ and is described by one and two body potentials

$$V_1(\phi) = 0, \quad V_2(\phi, \theta) = -\log\big|\mathrm{e}^{\mathrm{i}\phi} - \mathrm{e}^{\mathrm{i}\theta}\big|. \tag{4.3}$$

For the spectral density ρ°

$$\rho^\circ(\phi;\tau) = \frac{1}{N}\left\langle \sum_{i=1}^{N} \delta\big(\phi - \phi_i\big) \right\rangle_{P_\tau} \tag{4.4}$$

in the global regime where $\hat{\tau} = N\tau$, the hydrodynamical equation (1.9) is equal to

$$\frac{\partial \rho^\circ\big(\phi;\hat{\tau}\big)}{\partial \hat{\tau}} = -\frac{\partial}{\partial \phi}\left[\rho^\circ\big(\phi;\hat{\tau}\big) \frac{\partial}{\partial \phi}\left(\int_{-\pi}^{\pi} d\phi' \, \log\left| \mathrm{e}^{\mathrm{i}\phi} - \mathrm{e}^{\mathrm{i}\phi'} \right| \rho^\circ\big(\phi';\hat{\tau}\big) \right) \right]. \tag{4.5}$$

We define a circular Hilbert transform as

$$\mathcal{H}_\circ[f](\phi) := \frac{1}{2}\mathrm{PV} \int_I \mathrm{d}\phi' \, \cot\left(\frac{\phi - \phi'}{2} \right) f\big(\phi'\big), \qquad \phi \in \bar{I}, \tag{4.6}$$

where $\bar{I} \subset (-\pi, \pi]$ and, since $\partial_x \log|\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{\mathrm{i}y}| = \frac{1}{2}\cot\left(\frac{x-y}{2}\right)$, the equation (4.5) is expressed as

$$\frac{\partial \rho^\circ\big(\phi;\hat{\tau}\big)}{\partial \hat{\tau}} = -\frac{\partial}{\partial \phi}\Big(\rho^\circ\big(\phi;\hat{\tau}\big) \mathcal{H}_\circ\big[\rho^\circ(\,\cdot\,;\hat{\tau}) \big](\phi) \Big). \tag{4.7}$$

To arrive at the final evolution equation, we introduce a circular Green's function

$$G^\circ\big(z;\hat{\tau}\big) = \frac{1}{2}\int_{\bar{I}} \cot\left(\frac{z-y}{2} \right) \rho^\circ\big(y;\hat{\tau}\big)\, \mathrm{d}y, \tag{4.8}$$

which also satisfies (2.6) and (2.15) with replacements $G \mapsto G^\circ$ and $\rho \mapsto \rho^\circ$. Based on these properties, we again find the complex Burger's equation

$$\frac{\partial}{\partial \hat{\tau}} G^\circ\big(z;\hat{\tau}\big) + G^\circ\big(z;\hat{\tau}\big) \frac{\partial}{\partial z} G^\circ\big(z;\hat{\tau}\big) = 0, \tag{4.9}$$

which is formally in the same form as the Gaussian case (2.16). We can thus apply the same techniques to conclude that the solution of the initial value problem for this equation—initial spectral density $\rho^{\circ,(0)}$—reads

$$G^\circ\big(z;\hat{\tau}\big) = \int_{\bar{I}^{(0)}} \rho^{\circ,(0)}(\mu) \cot\left(\frac{z - \hat{\tau} G^\circ\big(z;\hat{\tau}\big) - \mu}{2} \right) \mathrm{d}\mu, \tag{4.10}$$

where $\bar{I}^{(0)}$ is the initial support of $\rho^{\circ,(0)}$. The hydrodynamical equation (4.5) was first derived by Pandey and Shukla [8, equation (59)], using the hierarchy of equations satisfied by the dynamical correlation functions. The general solution (4.10) is given in [8, equation (63)]. Our main point here is therefore not a new result, but rather a common theme, namely the macroscopic hydrodynamical equation (1.9).

The particular case $\rho^{\circ,(0)}(\mu) = \delta(\mu)$ was studied in the context of two-dimensional QCD by [23–25]. Even though there is no closed form solution of (4.10), several analytic features can be exhibited, including an effect analogous to that of the Gaussian ensemble evolution with initial condition (2.25): at a critical value to $\hat{\tau}$ two spectrum edges collide here being the left and right edges of the single interval of support.

4.2. Jacobi ensembles in trigonometric variables

We now move to the example of Jacobi ensembles. Consider a unitary (symmetric for $\beta = 1$, unconstrained by $\beta = 2$ or self dual for $\beta = 4$) matrix S of size $(n+m) \times (n+m)$ with $n \geqslant m$, divide it into 4 blocks

$$S = \begin{pmatrix} r_{n\times n} & t'_{n\times m} \\ t_{m\times n} & r'_{m\times m} \end{pmatrix}, \tag{4.11}$$

and investigate singular values of sub-block t'. The corresponding Jacobian of this ensemble (see [3, section 11.2.3]) reads

$$J = \prod_{j=1}^{m} \left(\lambda_j^2\right)^{\frac{\beta a'}{2}} \prod_{1\leqslant i<j\leqslant m} |\lambda_i^2 - \lambda_j^2|^{\beta}, \tag{4.12}$$

where $a' = n - m + 1 - \frac{1}{\beta}$ and $\lambda_i \in (0, 1)$ denote non-zero singular values of t'.

To obtain a Smoluchowski–Fokker–Planck equation for the joint PDF, we introduce new variables $\lambda_i = \sin\frac{\phi_i}{2}$, with $\phi_i \in (0, \pi)$. As was demonstrated in [3, section 11.2.3], the new variables $\{\phi_i\}$ permit an evolution (2.5) with a drift term

$$W = -\frac{a'}{2}\sum_{i=1}^{m}\log\sin^2\frac{\phi_i}{2} - \frac{b'}{2}\sum_{i=1}^{m}\log\cos^2\frac{\phi_i}{2} - \sum_{1\leqslant j<k\leqslant m}\log\left|\sin^2\frac{\phi_j}{2} - \sin^2\frac{\phi_k}{2}\right|, \tag{4.13}$$

with $b' = \frac{1}{\beta}$ and replacements $\{\lambda_i\} \to \{\phi_i\}$, $N \to m$. When compared to the Jacobian (4.12), an extra b' term arise by transforming the measure $\mathrm{d}\lambda_i = \frac{1}{2}\cos\frac{\phi_i}{2}\mathrm{d}\phi_i$. We read off one- and two body interactions from (4.13),

$$V_1(\phi) = -\frac{a'}{2}\log\sin^2\frac{\phi}{2} - \frac{b'}{2}\log\cos^2\frac{\phi}{2}, \qquad V_2\left(\phi, \phi'\right) = -\log\left|\sin^2\frac{\phi}{2} - \sin^2\frac{\phi'}{2}\right|,$$

and rewrite the latter

$$V_2\left(\phi, \phi'\right) = -\ln\left|\sin\left(\frac{\phi - \phi'}{2}\right)\right| - \ln\left|\sin\left(\frac{\phi + \phi'}{2}\right)\right|, \tag{4.14}$$

so that the $\phi \to -\phi$ symmetry is evident. Accordingly, we form a spectral density of the form

$$\rho^{\mathrm{J}}(\phi; \tau) = \frac{1}{m}\left\langle \sum_{i=1}^{m}\left(\delta\left(\phi - \phi_i\right) + \delta\left(\phi + \phi_i\right)\right)\right\rangle_{P_\tau}, \tag{4.15}$$

normalized to unity when integrated over $\phi \in (0, \pi)$ and even in ϕ. Both the interaction term and spectral density has features present in the chiral spectral density ρ^{c} (3.6) and the two body potential term (3.5). The current (1.9) driving the time evolution of ρ^{J} in the large m limit reads

$$\begin{aligned} J_{\mathrm{J}}\left(\phi; \hat{\tau}\right) &= \rho^{\mathrm{J}}\left(\phi; \hat{\tau}\right)\frac{\partial}{\partial\phi}\left(\frac{\hat{a}}{2}\log\sin^2\frac{\phi}{2} + \int_0^{\pi}\mathrm{d}\phi'\log\left|\sin^2\frac{\phi}{2} - \sin^2\frac{\phi'}{2}\right|\rho^{\mathrm{J}}\left(\phi'; \hat{\tau}\right)\right) \\ &= \rho^{J}\left(\phi; \hat{\tau}\right)\left(\frac{\hat{a}}{2}\cot\frac{\phi}{2} + \frac{1}{2}\int_{-\pi}^{\pi}\mathrm{d}\phi'\cot\frac{\phi - \phi'}{2}\rho^{J}\left(\phi'; \hat{\tau}\right)\right), \end{aligned} \tag{4.16}$$

where the $\hat{a} = \frac{n}{m} - 1, b'$ term has dropped out as subleading in the large m limit and the rescaled time parameter reads $\hat{\tau} = m\tau$. The hydrodynamic equation

$$\frac{\partial}{\partial\hat{\tau}}\rho^{\mathrm{J}}(\phi;\,\hat{\tau})-\frac{\partial}{\partial\phi}J_{\mathrm{J}}(\phi;\,\hat{\tau})$$

reads

$$\frac{\partial}{\partial\hat{\tau}}\rho^{\mathrm{J}}\big(\phi;\,\hat{\tau}\big)=-\frac{\partial}{\partial\phi}\Big(\rho^{\mathrm{J}}\big(\phi;\,\hat{\tau}\big)\mathcal{H}_{\circ}\Big[\hat{a}\delta(\,\cdot\,)+\rho^{\mathrm{J}}\big(\,\cdot\,;\,\hat{\tau}\big)\Big](\phi)\Big),\tag{4.17}$$

where the Hilbert transform $\mathcal{H}_{\circ}$ was already defined in (4.6). By using the properties of Green's function

$$G^{\mathrm{J}}\big(z;\,\hat{\tau}\big)=\frac{1}{2}\int_{\bar{I}_{+}}\left(\cot\left(\frac{z-y}{2}\right)+\cot\left(\frac{z+y}{2}\right)\right)\rho^{\circ}\big(y;\,\hat{\tau}\big)\,\mathrm{d}y,\tag{4.18}$$

with $\bar{I}_{+}\subset(0,\,\pi]$, we repeat the derivation of the complex Burgers equation (4.9) and obtain

$$\frac{\partial}{\partial\hat{\tau}}G^{\mathrm{J}}\big(z;\,\hat{\tau}\big)+\left(\frac{\hat{a}}{2}\cot\frac{z}{2}+G^{\mathrm{J}}\big(z;\,\hat{\tau}\big)\right)\frac{\partial}{\partial z}G^{\mathrm{J}}\big(z;\,\hat{\tau}\big)-\frac{\hat{a}}{4}\frac{G^{\mathrm{J}}\big(z;\,\hat{\tau}\big)}{\sin^{2}z/2}=0.\tag{4.19}$$

The equation has the same structure as the chiral Gaussian equation (3.12), and in fact reduces to that equation for small z. The underlying log-gas setup has therefore the same features—it consists of a fixed particle at $\phi=0$ of charge $\hat{a}$ and two mirror-like clouds for $\phi\in(-\pi,\,0)$ and $\phi\in(0,\,\pi)$ respectively. In the special case of vanishing charge $\hat{a}=0$, the resulting equation (4.19) coincides exactly with (4.9) obtained for the circular ensembles.

5. Collective variables

Collective variables is another approach to obtain the hydrodynamic equations (2.16) and (3.12). The idea of collective variables was first introduced in plasma physics [26] and extensively applied to gauge theories [27] and quantum Hall effect [28]. Besides re-deriving the aforementioned hydrodynamical equations, this method is suitable for obtaining asymptotic formulas for group integrals of Harish-Chandra/Itzykson–Zuber and Berezin–Karpelevich type. The former relate to the Gaussian ensembles whereas the latter appear in the chiral Gaussian ensembles. This relationship is the reason why we focus only on these two cases in this section, and don't consider the circular or Jacobi spectral evolutions.

5.1. Collective variables method

In the present context, one proceeds by transforming the Smoluchowski–Fokker–Planck equation (2.5) to new 'collective' type variables $\hat{\lambda}$,

$$\lambda_i\to\hat{\lambda}_j(\{\lambda\}),\qquad\{\lambda\}=\big(\lambda_1,\ldots,\lambda_N\big),\tag{5.1}$$

where $i=1\ldots N$, $j=1\ldots N'$. These new degrees of freedom should (a) use the symmetries of the system and (b) have a well defined large N limit. Typically $N'\to\infty$ from the beginning, and thus the particle system is treated as a fluid, so that the change is not bijective at least before taking the large N limit. For the special case $N'=N$ and N finite, the method corresponds to a bona fide variable change and was recently studied in the present context in [29]. The non-uniqueness of the collective variables means the aim is not an exact description in all regimes. However, since the new degrees of freedom conserve the symmetries, one expects to correctly reproduce certain macroscopic properties.

Consider a general transformation $\hat{\lambda}(q;\{\lambda\})$ with i index promoted to a variable q (i.e. $N' \to \infty$) in a fluid approximation. This continuous case introduces functional analysis by which the transformed Smoluchowski–Fokker–Planck equation

$$\partial_\tau \pi_\tau(\{\lambda\}) = L(\{\lambda\})\pi_\tau(\{\lambda\}), \qquad L(\{\lambda\}) = \frac{1}{\beta}\sum_{i=1}^{N}\frac{\partial^2}{\partial {\lambda_i}^2} - \sum_{i=1}^{N}\frac{\partial W}{\partial \lambda_i}\frac{\partial}{\partial \lambda_i}, \tag{5.2}$$

obtained by writing $p_\tau = \exp(-\beta W)\pi_\tau$ in (2.5), is transformed to a functional differential equation.

According to (5.1), the function of $\{\lambda\}$ becomes a functional in the q variables $\pi_\tau(\{\lambda\}) = \hat{\pi}_\tau[\hat{\lambda}(q;\{\lambda\})]$. Moreover, the Laplace–Beltrami operator L is re-expressed by the appropriate continuous chain rule

$$\frac{\partial}{\partial \lambda_j} = \sum_{i=1}^{N'}\frac{\partial \hat{\lambda}_i(\{\lambda\})}{\partial \lambda_j}\frac{\partial}{\partial \hat{\lambda}_i} \qquad \overset{'N'\to\infty'}{\longrightarrow} \qquad \frac{\partial}{\partial \lambda_j} = \int \mathrm{d}q \frac{\partial \hat{\lambda}(q;\{\lambda\})}{\partial \lambda_j}\frac{\delta}{\delta \hat{\lambda}(q)}, \tag{5.3}$$

where the $\hat{\lambda}(q)$ are the new variables just as $\hat{\lambda}_i$ in the discrete case. The transformed operator $\hat{L} = \hat{K} + \hat{V}$ reads

$$\hat{K} = \frac{1}{\beta}\int \mathrm{d}q \sum_{i=1}^{N}\frac{\partial^2 \hat{\lambda}(q)}{\partial \lambda_i^2}\frac{\delta}{\delta \hat{\lambda}(q)} + \frac{1}{\beta}\int \mathrm{d}p\mathrm{d}q \sum_{i=1}^{N}\frac{\partial \hat{\lambda}(q)}{\partial \lambda_i}\frac{\partial \hat{\lambda}(p)}{\partial \lambda_i}\frac{\delta^2}{\delta \hat{\lambda}(p)\delta \hat{\lambda}(q)}, \tag{5.4}$$

$$\hat{V} = -\int \mathrm{d}q \left(\sum_i \frac{\partial W}{\partial \lambda_i}\frac{\partial \hat{\lambda}(q)}{\partial \lambda_i}\right)\frac{\delta}{\delta \hat{\lambda}(q)}, \tag{5.5}$$

where we supressed the $\{\lambda\}$ dependence in the coefficients. The transformed Smoluchowski–Fokker–Planck equation (5.2) is then

$$\partial_\tau \hat{\pi}_\tau\big[\hat{\lambda}\big] = \Big(\hat{K}\big[\hat{\lambda}\big] + \hat{V}\big[\hat{\lambda}\big]\Big)\hat{\pi}_\tau\big[\hat{\lambda}\big]. \tag{5.6}$$

5.2. Gaussian ensembles

In the case of Gaussian ensembles, the drift term W is given by (2.6) and the collective variable

$$\hat{\lambda}(q;\{\lambda\}) = \sum_{i=1}^{N}\delta(q-\lambda_i) \tag{5.7}$$

is the (non-averaged) one-point correlation function (see (2.7)). This choice is consistent with condition (a) mentioned in the introduction to this section—it conserves the eigenvalue exchange symmetry. We calculate the kinetic part $\hat{K}$ with the help of the formula $\partial_{\lambda_i}\hat{\lambda} = -\partial_q \delta(q-\lambda_i)$, and the potential part $\hat{V}$ using

$$\frac{1}{\lambda_i - \lambda_j} = \mathrm{PV}\int_I \mathrm{d}\mu\, \frac{1}{\lambda_i - \mu}\delta(\lambda_j - \mu),$$
$$\sum_{i\neq j}\delta(p-\lambda_i)\delta(q-\lambda_j) = \hat{\lambda}(p)\hat{\lambda}(q) - \delta(p-q)\hat{\lambda}(p).$$

We set an ansatz for the leading large N form of the joint PDF

$$\hat{\pi}_\tau = \exp\left(-\frac{\beta}{2}N^2 S_\tau\right), \tag{5.8}$$

we find that the new functional S_τ satisfies the evolution equation

$$\partial_\tau S_\tau = \int \mathrm{d}p\ \hat{\lambda}(p)\left(\frac{1}{\beta}\frac{\partial^2}{\partial p^2}\left(\frac{\delta S_\tau}{\delta\hat{\lambda}(p)} + \frac{\delta^2 S_\tau}{\delta\hat{\lambda}(p)^2}\right) - \mathcal{H}[\delta](0)\frac{\partial}{\partial p}\frac{\delta S_\tau}{\delta\hat{\lambda}(p)}\right) +$$
$$- \int \mathrm{d}p\ \hat{\lambda}(p)\left(\frac{N^2}{2}\left(\frac{\partial}{\partial p}\frac{\delta S_\tau}{\delta\hat{\lambda}(p)}\right)^2 - \mathcal{H}\Big[\hat{\lambda}\Big](p)\frac{\partial}{\partial p}\frac{\delta S_\tau}{\delta\hat{\lambda}(p)}\right), \tag{5.9}$$

where $\mathcal{H}[f]$ denotes the Hilbert transform (2.11) with suppressed argument.

We perform the large N limit of (5.9) by rescaling both the time $N\tau = \hat{\tau}$ and the collective variable $\hat{\lambda} = N\rho$. In this limit, the first term on the rhs is subleading in N in comparison to the second and the time derivative on lhs. Ignoring this term, we obtain an equation for $S_{\hat{\tau}}$ in the Hamilton–Jacobi form,

$$\partial_{\hat{\tau}} S_{\hat{\tau}} + \int \mathrm{d}p\ \rho(p)\left[\frac{1}{2}\left(\frac{\partial}{\partial p}\frac{\delta S_{\hat{\tau}}}{\delta\rho(p)}\right)^2 - \mathcal{H}[\rho](p)\frac{\partial}{\partial p}\frac{\delta S_{\hat{\tau}}}{\delta\rho(p)}\right] = 0, \tag{5.10}$$

where the position variable is $\rho(p)$ and the conjugate momentum reads $\Delta(p) = \dfrac{\delta S_{\hat{\tau}}}{\delta\rho(p)}$. This allows $S_{\hat{\tau}}$ to be interpreted as an action evaluated on a physical trajectory between $\rho(p;\hat{\tau}=0)$ and $\rho(p;\hat{\tau})$. The resulting Hamiltonian $H = \int \mathrm{d}p\ \rho\left(\frac{1}{2}(\partial_p\Delta)^2 - \mathcal{H}[\rho]\partial_p\Delta\right)$ needs a minor reformulation since it contains a problematic Hilbert transform term. To this end, we invoke a canonical change of variables $(\rho, \Delta) \to (\rho', \Delta' = \Delta + C)$ with C dependent only on ρ. This change leaves the Hamiltonian unaltered i.e. $H'[\rho', \Delta] = H[\rho', \Delta' - C[\rho']]$ and the action picks up a boundary term

$$S'_{\hat{\tau}} = S_{\hat{\tau}} + T_{|\hat{\tau}} - T_{|0}, \tag{5.11}$$

where the subscripts denote boundary terms evaluated at initial $\hat{\tau} = 0$ and final time $\hat{\tau}$. The generating function T is found to be

$$T = -\frac{1}{2}\int \mathrm{d}p\mathrm{d}q\ \rho(p)\rho(q)\ln|p-q|, \quad \frac{\delta T}{\delta\rho(p)} = C. \tag{5.12}$$

The transformed Hamiltonian H' is

$$H'\Big[\rho', \Delta'\Big] = \frac{1}{2}\int \mathrm{d}p\ \rho'(p)\Big[\big(\partial_p\Delta'(p)\big)^2 - \Big(\mathcal{H}\Big[\rho'\Big](p)\Big)^2\Big] \tag{5.13}$$

for which the second term is re-expressed in term of ρ as

$$\int \mathrm{d}p\ \rho'(p)\Big(\mathcal{H}\Big[\rho'\Big](p)\Big)^2 = \frac{\pi^2}{3}\int \mathrm{d}p\ \rho'(p)^3. \tag{5.14}$$

This identity is proved using the properties of the Hilbert transform

$$\int f\mathcal{H}[g] = -\int g\mathcal{H}[f], \qquad 2\mathcal{H}[f\mathcal{H}[f]] = (\mathcal{H}[f])^2 - \pi^2 f^2,$$

valid for sufficiently well-behaved functions f, g [30]. From now on we drop the prime indices and the Hamiltonian (5.13) is finally

$$H[\rho, \Delta] = \frac{1}{2}\int \mathrm{d}p\ \rho\left(\big(\partial_p\Delta\big)^2 - \frac{\pi^2}{3}\rho^2\right), \tag{5.15}$$

with corresponding action

$$S_{\hat{\tau}} = \frac{1}{2}\int_0^{\hat{\tau}} \mathrm{d}\hat{t} \int \mathrm{d}p\ \rho\left(\left(\partial_p \Delta\right)^2 + \frac{\pi^2}{3}\rho^2 \right), \tag{5.16}$$

chosen so that $S_{\hat{\tau}}\,|_{\hat{\tau}=0} = 0$ (otherwise $S_{\hat{\tau}}$ is unique only up to an additive constant). By the Hamilton equations $\partial_{\hat{\tau}}\rho = \dfrac{\delta H}{\delta \Delta}$, $\partial_{\hat{\tau}}\Delta = -\dfrac{\delta H}{\delta \rho}$, the equations of motion read

$$\begin{aligned} &\partial_{\hat{\tau}}\Delta + \frac{1}{2}\left(\partial_p \Delta\right)^2 = \frac{\pi^2}{2}\rho^2, \\ &\partial_{\hat{\tau}}\rho + \partial_p\left(\rho \partial_p \Delta\right) = 0. \end{aligned} \tag{5.17}$$

Upon defining $G_{\pm} = \mp \mathrm{i}\pi\rho + \partial_p \Delta$ (see (2.14)), these formulas are exactly the complex Burger's equation (2.16). The construction goes similarly as before—equations (2.15) for $G_{\pm}$ are defined on the real line and induce a complex structure due to analytic properties of G.

This is a well-known result of Matytsin [31], reproduced also by other authors [32, 33]. Here we show how additionally the joint PDF function $\hat{\pi}$ is asymptotically expressed in terms of an action related to the hydrodynamical system.

5.3. Chiral Gaussian ensembles

For the chiral case we make suitable replacements (3.3) and W is defined in (3.4). The collective variable in this case is

$$\hat{x}(q,\{x\}) = \sum_{i=1}^{m} \delta(q - x_i) + \delta(q + x_i) = \sum_{i=1}^{m} 2\,|q|\,\delta\left(q^2 - x_i^2\right), \tag{5.18}$$

a (non-averaged) one-point correlation function (see (3.6)). Our task is therefore to transform (5.2). Because the derivation is parallel to the Gaussian case, we give only some partial results. In calculating the transformed Laplace operator $\hat{L}$, we use formulas

$$\begin{aligned} \frac{\partial}{\partial x_j}\left(\sum_{i=1}^{m} 2\,|q|\,\delta\left(q^2 - x_i^2\right) \right) &= -\frac{|q|}{q}\frac{\partial}{\partial q}\left(2\,|q|\,\delta\left(q^2 - x_j^2\right) \right), \\ \frac{1}{x_j}\delta\left(q^2 - x_j^2\right) &= \mathrm{PV}\int_{-\infty}^{\infty} \frac{\mathrm{d}\mu\ \delta(\mu)}{|q| - \mu}\delta\left(q^2 - x_j^2\right), \\ \frac{2x_j}{x_j^2 - x_k^2}\delta\left(q^2 - x_j^2\right) &= \mathrm{PV}\int_{-\infty}^{\infty} \frac{\mathrm{d}\mu 2\,|\mu|\,\delta\left(\mu^2 - x_k^2\right)}{|q| - \mu}\delta\left(q^2 - x_j^2\right). \end{aligned}$$

We make a large m joint PDF ansatz

$$\hat{\pi}_{\tau} = \exp\left(-\frac{\beta}{4} m^2 S_{\tau}^{\mathrm{c}} \right) \tag{5.19}$$

which captures the rough degrees of freedom $\sim m^2$ and trivial β dependence. The equation satisfied by S_{τ}^{c} reads

$$\partial_\tau S_\tau^{\rm c} = \int {\rm d}p\ \hat{x}(p)\left(\frac{1}{\beta}\frac{\partial^2}{\partial p^2}\left(\frac{\delta S_\tau^{\rm c}}{\delta \hat{x}(p)} + 2\frac{\delta^2 S_\tau^{\rm c}}{\delta \hat{x}(q)^2}\right) + \mathcal{H}[\delta(.\,)](0)\frac{\partial}{\partial p}\frac{\delta S_\tau^{\rm c}}{\delta \hat{x}(p)}\right) +$$
$$- \int {\rm d}p\ \hat{x}(p)\left(\frac{m^2}{2}\left(\frac{\partial}{\partial p}\frac{\delta S_\tau^{\rm c}}{\delta \hat{x}(p)}\right)^2 - \mathcal{H}\big[a'\delta(.\,) + \hat{x}(.\,)\big](p)\frac{\partial}{\partial p}\frac{\delta S_\tau^{\rm c}}{\delta \hat{x}(p)}\right). \tag{5.20}$$

Now we perform a $m, n \to \infty$ limit with n/m fixed. We set $\hat{x} = m\rho^{\rm c}$, $\hat{\tau} = m\tau$ and find the first term on rhs subleading w.r.t. the second and the time derivative. The equation for $S_{\hat{\tau}}^{\rm c}$ is again in the Hamilton–Jacobi form

$$\partial_{\hat{\tau}} S_{\hat{\tau}}^{\rm c} + \int {\rm d}p\ \rho^{\rm c}(p)\left(\frac{1}{2}\left(\frac{\partial}{\partial p}\frac{\delta S_{\hat{\tau}}^{\rm c}}{\delta \rho^{\rm c}(p)}\right)^2 - \frac{a'}{p}\frac{\partial}{\partial p}\frac{\delta S_{\hat{\tau}}^{\rm c}}{\delta \rho^{\rm c}(p)} - \mathcal{H}\big[\rho^{\rm c}\big](p)\frac{\partial}{\partial p}\frac{\delta S_{\hat{\tau}}^{\rm c}}{\delta \rho^{\rm c}(p)}\right) = 0. \tag{5.21}$$

With the conjugate momentum $\Delta^{\rm c} = \dfrac{\delta S_{\hat{\tau}}^{\rm c}}{\delta \rho^{\rm c}}$, we transform $(\rho^{\rm c}, \Delta^{\rm c}) \to ((\rho^{\rm c})' = \rho^{\rm c}, (\Delta^{\rm c})' = \Delta^{\rm c} + C^{\rm c})$ and the action is transformed accordingly

$$\big(S_{\hat{\tau}}^{\rm c}\big)' = S_{\hat{\tau}}^{\rm c} + T^{\rm c}_{|\hat{\tau}} - T^{\rm c}_{|0} \tag{5.22}$$

with a generating function

$$T^{\rm c} = -\frac{1}{2}\int_{-\infty}^{\infty} {\rm d}p{\rm d}q\ \log|p-q|\rho^{\rm c}(p)\rho^{\rm c}(q), \qquad \frac{\delta T^{\rm c}}{\delta \rho^{\rm c}} = C^{\rm c}. \tag{5.23}$$

Under this transformation, the new Hamiltonian reads

$$H\big[\rho^{\rm c}, \Delta^{\rm c}\big] = \frac{1}{2}\int {\rm d}p\ \rho^{\rm c}(p)\left(\left(\frac{\partial}{\partial p}\frac{\delta S_{\hat{\tau}}^{\rm c}}{\delta \rho^{\rm c}(p)}\right)^2 - \big(\mathcal{H}\big[\rho^{\rm c}\big](p)\big)^2 + \frac{2\hat{a}}{p}\left(\frac{\partial}{\partial p}\frac{\delta S_{\hat{\tau}}^{\rm c}}{\delta \rho^{\rm c}(p)} - \mathcal{H}\big[\rho^{\rm c}\big](p)\right)\right), \tag{5.24}$$

where $\hat{a} = \dfrac{n}{m} - 1$ and we dropped the primed indices. As before, the term quadratic in Hilbert transforms is given in (5.14) whereas the linear term vanishes by two additional properties of $\mathcal{H}$ [30],

$$\mathcal{H}\left[\frac{f(.\,)}{(.\,)}\right](x) = \frac{\mathcal{H}[f]}{x} - \frac{\mathcal{H}[f](0)}{x}, \tag{5.25}$$

$$\mathcal{H}[f](0) = 0, \quad \text{if} \quad f(x) = f(-x). \tag{5.26}$$

Thus the final form of the Hamiltonian reads

$$H\big[\rho^{\rm c}, \Delta^{\rm c}\big] = \frac{1}{2}\int {\rm d}p\ \rho^{\rm c}\left(\big(\partial_p\Delta^{\rm c}\big)^2 - \frac{\pi^2}{3}\big(\rho^{\rm c}\big)^2 + \frac{2\hat{a}}{p}\partial_p\Delta^{\rm c}\right), \tag{5.27}$$

and the action is therefore

$$S_{\hat{\tau}}^{\rm c} = \frac{1}{2}\int_0^{\hat{\tau}} {\rm d}\hat{t}\int {\rm d}p\ \rho^{\rm c}\left(\big(\partial_p\Delta^{\rm c}\big)^2 + \frac{\pi^2}{3}\big(\rho^{\rm c}\big)^2 + \frac{2\hat{a}}{p}\partial_p\Delta^{\rm c}\right). \tag{5.28}$$

The Hamilton equations of motion read

$$\partial_{\hat{\tau}}\Delta^{\mathrm{c}} + \frac{1}{2}\big(\partial_p\Delta^{\mathrm{c}}\big)^2 + \frac{\hat{a}}{p}\partial_p\Delta^{\mathrm{c}} = \frac{\pi^2}{2}\big(\rho^{\mathrm{c}}\big)^2,$$
$$\partial_{\hat{\tau}}\rho^{\mathrm{c}} + \partial_p\big(\rho^{\mathrm{c}}\partial_p\Delta^{\mathrm{c}}\big) + \partial_p\left(\frac{\hat{a}}{p}\rho^{\mathrm{c}}\right) = 0. \tag{5.29}$$

We observe again how the chiral case reduces to Gaussian (2.16) when $\hat{a} = 0$, and by defining $G^{\mathrm{c}}_{\pm} = \mp\mathrm{i}\pi\rho^{\mathrm{c}} + \partial_p\Delta^{\mathrm{c}}$ we reclaim (3.12) by the arguments elucidated previously.

6. Asymptotic expansion of Harish-Chandra/Itzykson–Zuber and Berezin–Karpelevich integrals

Collective variables were used by Matytsin [31] to obtain large N expansion of the celebrated Harish-Chandra/Itzykson–Zuber integral formula. Later works looked at the same task from both mathematical [32] and physical point of view [33]. In this section we comment on this standard result and afterwards use analogous working to compute an expansion for the Berezin–Karpelevich type integrals [17, 34] arising in the chiral Gaussian ensembles.

6.1. HCIZ-type integrals

We consider an integral

$$I_\beta(A, B) = \int \big(U^\dagger \mathrm{d}U\big)\exp\left(\frac{\beta N}{2}\,\mathrm{Tr}\big(UAU^\dagger B\big)\right), \tag{6.1}$$

where matrices A, B are diagonal of size $N \times N$ and U are real orthogonal ($\beta = 1$) or complex unitary ($\beta = 2$). In the RMT context these integrals arise in connection with the Gaussian ensembles. For $\beta = 2$ an exact formula exists, found independently by Harish-chandra [35] and Itzykson–Zuber [36].

To obtain large N asymptotic behavior of (6.1), we recall the definition (2.1) of joint PDF

$$\pi_{\hat{\tau}} = \int \big(U^\dagger \mathrm{d}U\big)P_{\hat{\tau}}\big(X^{(0)};\, ULU^\dagger\big) = \frac{1}{C_{N,\hat{\tau}}}\mathrm{e}^{-\frac{\beta N}{4\hat{\tau}}\,\mathrm{Tr}\,L^2 - \frac{\beta N}{4\hat{\tau}}\,\mathrm{Tr}\left(X^{(0)}\right)^2} I_\beta\left(\frac{L}{\sqrt{\hat{\tau}}}, \frac{X^{(0)}}{\sqrt{\hat{\tau}}}\right). \tag{6.2}$$

The traces in this expression can be given in term of the collective variable (5.7) as $\mathrm{Tr}\,L^2 = N\int \mathrm{d}p\; p^2\rho(p;\hat{\tau})$ and $\mathrm{Tr}(X^{(0)})^2 = N\int \mathrm{d}p\; p^2\rho(p;\hat{\tau}=0)$ in accordance to their role as an initial and final densities respectively. On the other hand, the asymptotic form of π_τ was found in section 5.2 as

$$\pi_{\hat{\tau}} \sim \exp\left(-\frac{\beta}{2}N^2\big(S_{\hat{\tau}} - T_{|\hat{\tau}} - T_{|0}\big)\right) \tag{6.3}$$

using (5.8) and (5.11), and where we also added an arbitrary constant to the action $S_{\hat{\tau}} \to S_{\hat{\tau}} - 2T_{|0}$ (recall the comment below (5.16)). The form of this constant is chosen so that $\pi_{\hat{\tau}}\,|_{\hat{\tau}=0} \sim \exp\big(\beta N^2 T_{|0}\big)$.

To arrive at an asymptotic expression for (6.1), we fix the time $\hat{\tau} = 1$ and rename the final $\rho(p, \hat{\tau} = 1) = \rho_f(p)$ and initial $\rho(p, \hat{\tau} = 0) = \rho_i(p)$ densities

$$I_\beta(\sigma,\,\alpha) \sim C_{N,\hat\tau=1}\exp\left(\frac{\beta}{2}N^2\Big[-S_{\hat\tau=1}+\frac{1}{2}\int \mathrm{d}p\; p^2\Big(\rho_i(p)+\rho_f(p)\Big)+ \right.$$
$$\left. +\frac{1}{2}\int \mathrm{d}p\mathrm{d}q\Big(\rho_i(p)\rho_i(q)+\rho_f(p)\rho_f(q)\Big)\ln|q-p|\Big]\right). \tag{6.4}$$

Now the main difficulty lies in finding a physical path joining initial $\rho_i(p)$ and final $\rho_f(p)$ spectral densities and calculating the corresponding action $S_{\hat\tau}$, which is specified by (5.16). The former problem has been solved from our workings in sections 1 and 2 for the initial conditions (1.6) and (2.25), while the evaluation of $S_{\hat\tau}$ in the first of these is given in [33]. For a discussion of analyticity properties of (6.4) see [37].

6.2. Berezin–Karpelevich type integrals

We now turn to the asymptotic formula for an integral of Berezin–Karpelevich type defined as

$$J_\beta(A,\,B)=\int\Big(U^\dagger\mathrm{d}U\Big)\Big(V^\dagger\mathrm{d}V\Big)\exp\left(\frac{\beta m}{4}\,\mathrm{Tr}\,\Big(VA^\dagger U^\dagger B+B^\dagger UAV^\dagger\Big)\right), \tag{6.5}$$

where A, B are $n\times m$ diagonal matrices and U, V are real orthogonal ($\beta=1$) or complex unitary ($\beta=2$) matrices of sizes $n\times n$ and $m\times m$ respectively with $n\geqslant m$. These integrals arise in studying chiral/Wishart/Laguerre type ensembles.

In the $\beta=2$ case, an exact formula was rediscovered in [34] and originally calculated by Berezin and Karpelevich [38]. To obtain an asymptotic expression for $\beta=1,\,2$ we recall the chiral joint PDF (3.2),

$$\pi_{\hat\tau}=\int\Big(U^\dagger\mathrm{d}U\Big)\Big(V^\dagger\mathrm{d}V\Big)P_{\hat\tau}\Big(Z^{(0)};\,ULV^\dagger\Big)$$
$$=\frac{1}{C^{\mathrm{c}}_{N,\hat\tau}}\mathrm{e}^{-\frac{\beta m}{4\hat\tau}\,\mathrm{Tr}\,\left(L^\dagger L+(Z^{(0)})^\dagger Z^{(0)}\right)}J_\beta\left(\frac{L}{\sqrt{\hat\tau}},\,\frac{Z^{(0)}}{\sqrt{\hat\tau}}\right). \tag{6.6}$$

We introduce the normalized collective variables to the Gaussian terms $\mathrm{Tr}\,L^\dagger L=\dfrac{m}{2}\int\mathrm{d}q\;q^2\rho^{\mathrm{c}}(q;\,\hat\tau)$ and $\mathrm{Tr}(Z^{(0)})^\dagger Z^{(0)}=\dfrac{m}{2}\int\mathrm{d}q\;\;q^2\rho^{\mathrm{c}}(q;\,\hat\tau=0)$. An asymptotic form of lhs was found in section 5.3 as

$$\pi_{\hat\tau}\sim\exp\left(-\frac{\beta m^2}{4}\Big(S^{\mathrm{c}}_{\hat\tau}-T^{\mathrm{c}}_{|\hat\tau}-T^{\mathrm{c}}_{|0}\Big)\right) \tag{6.7}$$

along with adding a constant $S^{\mathrm{c}}_{\hat\tau}\mapsto S^{\mathrm{c}}_{\hat\tau}-2\,T^{\mathrm{c}}_{|0}$. By comparing (6.6) and (6.7), for fixed time $\hat\tau=1$ we have an asymptotic formula

$$J_\beta\Big(\rho^{\mathrm{c}}_f,\,\rho^{\mathrm{c}}_f\Big)\sim C^{\mathrm{c}}_{N,\hat\tau=1}\exp\left(\frac{\beta}{4}m^2\Big[-S^{\mathrm{c}}_{\hat\tau=1}+\frac{1}{2}\int\mathrm{d}p\;p^2\Big(\rho^{\mathrm{c}}_i(p)+\rho^{\mathrm{c}}_f(p)\Big)+\right.$$
$$\left.+\frac{1}{2}\int\mathrm{d}p\mathrm{d}q\Big(\rho^{\mathrm{c}}_i(p)\rho^{\mathrm{c}}_i(q)+\rho^{\mathrm{c}}_f(p)\rho^{\mathrm{c}}_f(q)\Big)\ln|q-p|\Big]\right). \tag{6.8}$$

where initial and final densities are denoted as $\rho^{\mathrm{c}}(p;\,\hat\tau=0)=\rho^{\mathrm{c}}_i(p)$ and $\rho^{\mathrm{c}}(p;\,\hat\tau=1)=\rho^{\mathrm{c}}_f(p)$ respectively. As in the case of (6.4), we comment that to obtain the asymptotic formula for prescribed initial ρ^{c}_i and final ρ^{c}_f densities, it is necessary to evaluate the action S^{c}_τ on a physical trajectory connecting these two spectral densities. We note that such trajectories are given for particular initial conditions below (3.28).

Acknowledgments

The work of PJF was supported by the Australian Research Council discovery project grant DP140102613 and by the ARC Centre of Excellence for Mathematical and Statistical Frontiers. JG thanks Melbourne University for the warm hospitality during the preparation of this work and acknowledges the support of both the Grant DEC-2011/02/A/ST1/00119 of the National Centre of Science and the Australian Government Endeavour Fellowship.

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Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

Special Issue on Unsolved Problems of Noise in Physics, Biology and Technology

Ornstein–Uhlenbeck diffusion of hermitian and non-hermitian matrices—unexpected links

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Received 22 October 2015
Accepted for publication 4 January 2016
Published 20 May 2016

Online at stacks.iop.org/JSTAT/2016/054037
doi:10.1088/1742-5468/2016/05/054037

Abstract. We compare the Ornstein–Uhlenbeck process for the Gaussian unitary ensemble to its non-hermitian counterpart—for the complex Ginibre ensemble. We exploit the mathematical framework based on the generalized Green's functions, which involves a new, hidden complex variable, in comparison to the standard treatment of the resolvents. This new variable turns out to be crucial to understand the pattern of the evolution of non-hermitian systems. The new feature is the emergence of the coupling between the flow of eigenvalues and that of left/right eigenvectors. We analyze local and global equilibria for both systems. Finally, we highlight some unexpected links between both ensembles.

Keywords: disordered systems (theory), Brownian motion, random matrix theory and extensions, Schock waves

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1742-5468/16/054037+22$33.00

Contents

1. Introduction

In 1962, Dyson suggested an inspiring way to understand the joint probability distribution function (hereafter jpdf) of the eigenvalues of random matrices. In order to find it, he introduced an auxiliary dynamics in some fictitious 'time', which, in the large time limit, leads to a stationary state (Gibbs state) representing the desired jpdf. As he pointed out [1]: '*After considerable and fruitless efforts to develop a Newtonian theory of ensembles, we discovered that the correct procedure is quite different and much simpler. The* x_i [eigenvalues] *should be interpreted as positions of particles in Brownian motion*'. The resulting stationary distributions (originally for hermitian or for unitary random matrices) were obtained as a result of Ornstein–Uhlenbeck diffusion with a drift force coming from electrostatic-like repulsion of eigenvalues. The success of this description has contributed to multiple applications of random matrix models in practically all branches of science. The notion of 'time' has evolved as well, so nowadays it can be a physical parameter, representing either the real time or, e.g. the length of a mesoscopic wire, the area of a string or an external temperature. The idea of a noisy walk of eigenvalues recently led also to such concepts as determinantal processes [2–4], Loewner diffusion [5], fluctuations of non-intersecting interfaces in thermal equilibrium [6] and

doi:10.1088/1742-5468/2016/05/054037

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the emergence of pre-shock spectral waves and universal scaling at the critical points of several random matrix models.

Three years after Dyson, Ginibre [7] considered for the first time strictly non-hermitian random matrix models, whose spectrum does not need to be confined either to the real line (hermitian operators) or to the unit circle (unitary operators), but can be located on a two-dimensional support on the complex plane. The original motivation for the study of complex, random spectra was purely academic. Today however, non-hermitian random operators play a role in quantum information processing, in financial engineering (when lagged correlations are discussed [8]) or in identifying clusters in social or biological networks using non-backtracking operators [9], to name just a few recent applications. Additionally, statistical properties of eigenvectors of non-hermitian operators contribute to understanding scattering problems in open chaotic cavities [10] and random lasing.

It is surprising that in the last half century, the Dysonian picture of random walk of eigenvalues was not applied to the complex Ginibre ensemble (GE). The Brownian walk problem for the real GE was recently studied in [11]. In this contribution, partially based on our earlier work on this subject, we show how to fill this logical gap, and we also speculate on the reasons why the non-hermitian extension of a random walk scenario was far from obvious.

In section 2, we start from recalling Dyson's original construction [1]. Then, we propose an alternative description, where the fundamental object is the characteristic polynomial. We show the advantages of such description, borrowing heavily from the analogies to the simplest model of turbulence, i.e. the so-called Burgers equation. We also briefly mention, how the seminal results for the Gaussian unitary ensemble (GUE) can be recovered from a Burgers-like description.

In section 3, we formulate a mathematical framework, which allows us to parallel the turbulent picture in the case of the GE. In particular, we unravel a hidden dynamics associated with a new complex variable, which in standard descriptions of non-hermitian random matrix models is treated as an infinitesimal regulator only. We point out, that the non-hermitian character of the GE binds the dynamics of eigenvalues to the evolution of eigenvectors in a non-trivial way. Alike in the case of GUE, we demonstrate how the well-known results of the GE can be easily reclaimed in our formalism. Section 4 contains numerical experiments for both GUE and GE capturing the relevant diffusive degrees of freedom. In section 5, we uncover the unexpected links between the descriptions of the Gaussian unitary and the GEs. Section 6 concludes the paper and lists some open problems on noise in matrix models.

2. Diffusion in the Gaussian unitary ensemble

According to Dyson, the eigenvalues of a random, N by N hermitian matrix belonging to the GUE fulfill the following stochastic equation

$$\mathrm{d}\lambda_i(\tau) = \frac{1}{\sqrt{N}}\mathrm{d}B_i(\tau) + \frac{1}{N}\sum_{j=1,j\neq i}^{N}\frac{1}{\lambda_i-\lambda_j}\mathrm{d}\tau - a\lambda_i\mathrm{d}\tau, \tag{1}$$

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where B_is are one-dimensional standard Brownian motions, λ_is denote the eigenvalues and τ is the time variable. The second term represents a fictitious electric field coming from the logarithmic Coulomb potential (originating from the Van der Monde determinant) and the last term represents the drift coming from the confining harmonic potential (Ornstein–Uhlenbeck process). In the limit when N tends to infinity and $\tau \to \infty$, the eigenvalues freeze-out as a result of the compromise between the repulsion (electric field) and attraction (harmonic potential). The resulting spectral distribution takes the form of the Wigner semicircle. Despite the fact that Dyson was primarily interested in the equilibrium state, and introduced a time in an auxiliary construction, he pointed out that the transition to the equilibrium is quite subtle. In his own words [1], the Coulombic term is '*measuring the frequency with which two charges come into coincidence. This term is mainly sensitive to the local (microscopic) configurations of the gas particles... at the microscopic time scale ... After local equilibrium is established... the gas must adjust itself by macroscopic motion on the time scale*', which is N times larger compared to the microscopic one. He also noted that [1] '*a rigorous proof that this picture is accurate would require a much deeper mathematical analysis*'. The discussion in this section gives support to this picture.

Let us introduce an $N \times N$ hermitian matrix H by defining its complex entries according to:

$$H_{ij} = \begin{cases} x_{ii}, & i = j, \\ x_{ij} + \mathrm{i} y_{ij}, & i \neq j, \end{cases} \tag{2}$$

where $x_{ij} = x_{ji}$ and $y_{ij} = -y_{ji}$, with x_{ij} and y_{ij} real. Furthermore let x_{ij} and y_{ij} perform white noise driven, independent random walks, such that

$$\left\langle \delta H_{ij} \right\rangle = -a H_{ij} \delta\tau, \qquad \left\langle \left| \delta H_{ij} \right|^2 \right\rangle = \frac{g_{ij}}{N} \delta\tau, \tag{3}$$

with $g_{ij} = 1 + \delta_{ij}$ and for any i and j. Let $P(x_{ij}, \tau) P(y_{ij}, \tau)$ be the probability that the off diagonal matrix entry H_{ij} will change from its initial state to $x_{ij} + \mathrm{i} y_{ij}$ after time τ. Analogically, $P(x_{ii}, \tau)$ is the probability of the diagonal entry H_{ii} becoming equal to x_{ii} at τ. The evolution of these functions is governed by the following Smoluchowski–Fokker–Planck (SFP) equations:

$$\begin{aligned} \frac{\partial}{\partial\tau} P(x_{ii}, \tau) &= \left(\frac{1}{2N} \frac{\partial^2}{\partial x_{ii}^2} + a \frac{\partial}{\partial x_{ii}} x_{ii} \right) P(x_{ii}, \tau), \\ \frac{\partial}{\partial\tau} P(v_{ij}, \tau) &= \left(\frac{1}{4N} \frac{\partial^2}{\partial v_{ij}^2} + a \frac{\partial}{\partial v_{ij}} v_{ij} \right) P(v_{ij}, \tau), \quad i < j, \end{aligned} \tag{4}$$

where the parameter a measures the strength of the harmonic potential confining the diffusion of the matrix elements and v_{ij} denotes either x_{ij} or y_{ij}. The joint probability density function is thus defined as

$$P(x, y, \tau) \equiv \prod_{k=1}^{N} P(x_{kk}, \tau) \prod_{i<j=1}^{N} P(x_{ij}, \tau) P(y_{ij}, \tau) \tag{5}$$

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and satisfies the following equation

$$\partial_\tau P(x,y,\tau) = \mathcal{A}(x,y)P(x,y,\tau), \tag{6}$$

with

$$\mathcal{A}(x,y) = \sum_{k=1}^{N}\left(\frac{1}{2N}\frac{\partial^2}{\partial x_{kk}^2} + a\frac{\partial}{\partial x_{kk}}x_{kk}\right) + \frac{1}{4N}\sum_{i<j=1}^{N}\left(\frac{\partial^2}{\partial x_{ij}^2} + \frac{\partial^2}{\partial y_{ij}^2}\right) + a\sum_{i<j=1}^{N}\left(\frac{\partial}{\partial x_{ij}}x_{ij} + \frac{\partial}{\partial y_{ij}}y_{ij}\right). \tag{7}$$

A source-like solution of (6) reads:

$$P(x,y,\tau) = C\exp\left(-\frac{Na}{1-\mathrm{e}^{-2a\tau}}\mathrm{Tr}(H - H_0\mathrm{e}^{-a\tau})^2\right), \tag{8}$$

with $H(\tau=0) = H_0$ and C is a normalization constant. With the setting thus defined, let us proceed to the derivation of the partial differential equations obeyed by the averaged characteristic polynomial (hereafter called ACP) $U(z,\tau)$ associated with the diffusing matrix H:

$$U(z,\tau) \equiv \langle \det(z-H)\rangle_\tau, \tag{9}$$

where the angular brackets denote the averaging over the time dependent probability density (5). In appendix A, we show that the ACP satisfies

$$\partial_\tau U(z,\tau) = -\frac{1}{2N}\partial_{zz}U(z,\tau) + az\partial_z U(z,\tau) - aNU(z,\tau). \tag{10}$$

Note that the standard Green's function (resolvent) associated with H is related to ACP in the large N limit

$$G \equiv G(z,\tau) \equiv \lim_{N\to\infty}\frac{1}{N}\langle \mathrm{Tr}\frac{1}{z-H}\rangle = \lim_{N\to\infty}\frac{1}{N}\partial_z \ln U. \tag{11}$$

Thus, the spectral density of H is given by

$$\rho(\lambda,\tau) = -\frac{1}{\pi}\lim_{\epsilon\to 0_+}\mathrm{Im}\,G(z=\lambda+\mathrm{i}\epsilon,\tau), \tag{12}$$

through the Sokhotski–Plemelj formula.

Let us define $f_N \equiv f_N(z,\tau) \equiv \frac{1}{N}\partial_z \ln U$ (this is the complex analogue of the Cole–Hopf transform). Due to equation (10), f_N satisfies:

$$\partial_\tau f_N + f\partial_z f_N - a\partial_z(zf_N) = -\frac{1}{2N}\partial_{zz}f_N. \tag{13}$$

Since $\lim_{N\to\infty} f_N = G$, we see that the Green's function is governed by the following complex Burgers-like differential equation:

$$\partial_\tau G + G\partial_z G - a\partial_z(zG) = 0. \tag{14}$$

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2.1. Macroscopic equilibrium

In the $\tau \to \infty$ limit the time derivative in equation (14) vanishes and so this equation is easily solvable since it reduces to:

$$\partial_z\left(\frac{1}{2}G^2 - azG\right) = 0. \tag{15}$$

Because all moments $\frac{1}{N}\left\langle \mathrm{Tr}H^k \right\rangle$ are finite for any k, the function $G(z,\tau)$ has to tend to zero as $1/z$ in the $z \to \infty$ limit. This observation fixes the integration constant of equation (15) to be equal to $-a$. The resulting solution of the quadratic equation

$$\frac{1}{2}G(z)^2 - azG(z) + a = 0 \tag{16}$$

reads $G(z) = a(z - \sqrt{z^2 - 2/a})$. Using (12), for the standard value $a = 1/2$, we recover the Wigner semicircle

$$\rho(x) = \frac{1}{2\pi}\sqrt{4 - x^2}. \tag{17}$$

The analogy to the Burgers equation is however deeper, as we pointed out in [12]. For the real Burgers equation, the solution based on the method of characteristics breaks down due to the emergence of the pre-shock wave (singularity). Similar phenomenon takes place on the complex plane for equation (14). The resulting spectral shock waves correspond to the endpoints of the spectrum. Several other, surprising links between the Burgers equation and some probabilistic models are discussed in [13].

2.2. Microscopic limit

We stress here that equation (10) for the diffusing characteristic polynomial is exact for any N and for any initial conditions. We can therefore use it to retrieve the spectral features at all time scales and at all points of the spectrum. In order to simplify the analysis, let us start by performing a useful change of variables belonging to the class of Lamperti transformations [14–17]. This change of variables will allow to establish a connection between the Ornstein–Uhlenbeck process and the case of free diffusion ($a = 0$), where the results are known [12, 18]. To check that indeed we can recover a free diffusion equation from the Ornstein–Uhlenbeck process, we explicitly write down the relevant Lamperti transformation:

$$\begin{aligned} &U(z,\tau) = (1 + 2a\tau')^{-N/2}U'(z',\tau'), \\ &z' = \mathrm{e}^{a\tau}z, \quad \tau' = \frac{1}{2a}(\mathrm{e}^{2a\tau} - 1). \end{aligned} \tag{18}$$

Straightforward but lengthy calculations yield

$$\begin{aligned} &\partial_\tau U = A^{-2}\left(-\frac{Na}{1 + 2a\tau'}BU' + B\partial_{\tau'}U'\right) + az'B\partial_{z'}U', \\ &\partial_z U = BA^{-1}\partial_{z'}U', \qquad \partial_{zz}U = BA^{-2}\partial_{z'z'}U', \end{aligned}$$

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where we define $A = \mathrm{e}^{-a\tau}$ and $B = (1 + 2a\tau')^{-N/2}$. Plugging these into (10) produces a free diffusion equation in the variables (z', τ'):

$$\partial_{\tau'} U' = -\frac{1}{2N} \partial_{z'z'} U'. \tag{19}$$

In order to avoid unnecessary repetitions, we highlight here only the consequences of equation (19), relegating the details to already published work [12, 18]. First, it is exactly integrable on the complex plane for any initial conditions. The corresponding Cole–Hopf transformation maps the diffusion equation onto the complex viscid Burgers equation

$$\partial_{\tau'} f'_N + f' \partial_{z'} f'_N = -\nu_s \partial_{z'z'} f'_N, \tag{20}$$

where the (negative) spectral viscosity reads $\nu_s = 1/2N$. Second, in the large N limit (inviscid limit), spectral shock waves form at the endpoints of the spectra. Third, in the vicinity of the shock waves (endpoints of the spectra), the above equation captures the microscopic universality of the GUE ensembles, leading to Airy function oscillations at the edges. Intuitively, spectral oscillations at the endpoint origin from the negative sign of viscosity, which causes the 'roughening' of the transition instead of smoothening it, as would be expected in the case of a positive viscosity. Last but not least, the above picture confirms that Dysonian microscopic equilibrium is formed already at very short time scales and its character is determined solely by the global properties of the random matrix, i.e. the symmetries deciding on the functional form of the Coulombic repulsion.

3. Diffusion in the Ginibre ensemble

Contrary to the case of the hermitian ensembles, the spectrum of non-hermitian matrices is genuinely complex. Let us define the simplest example, the so-called complex GE where each element of the $N \times N$ matrix X is drawn from a complex Gaussian distribution. That is, each entry $X_{ij} = x_{ij} + \mathrm{i}y_{ij}$ consists of x_{ij} and y_{ij} drawn from standard Gaussian distributions. Note that all moments $\langle \mathrm{Tr} X^n \rangle$ (and $\langle \mathrm{Tr}(X^\dagger)^n \rangle$) vanish because $\langle \mathrm{Tr} X^2 \rangle = 0$. The only non vanishing moments are the mixed ones, i.e. $\langle \mathrm{Tr}(XX^\dagger)^n \rangle$. As we will show, in the large N limit, the eigenvalues condense uniformly on the centered disc on the complex plane. Therefore, the spectrum exhibits a jump at the rim, contrary to the hermitian cases, when the real spectrum is continuous at the endpoints, and only the derivatives of the spectrum are discontinuous. Moreover, the spectrum is non-analytic inside the disc, which seems to disqualify all the methods based on analyticity of the complex variable z. This is best visible when we try to repeat the hermitian construction for the Green's function $G(z) = \frac{1}{N}\left\langle \mathrm{Tr} \frac{1}{z - X} \right\rangle$. Since all moments vanish, such a Green's function is simply equal to $G(z) = \frac{1}{z}$, and does not reflect correctly the spectral properties of the ensemble. Similarly, the characteristic determinant is trivial, $\langle \det(z - X) \rangle = z^N$.

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The way out, based on an electrostatic analogy, was suggested a long time ago [21]. We define an electrostatic potential

$$\Phi \equiv \Phi(z) = \lim_{\epsilon\to 0}\lim_{N\to\infty}\frac{1}{N}\langle \mathrm{Tr}\ln[|z-X|^2+\epsilon^2]\rangle\ , \tag{21}$$

where we use a short-hand notation: $|z-X|^2+\epsilon^2 = (z\mathbf{1}_N - X)(\bar{z}\mathbf{1}_N - X^\dagger)+\epsilon^2\mathbf{1}_N$, where $\mathbf{1_N}$ is the N-dimensional identity matrix. Then, we calculate the 'electric field' as a gradient of the electrostatic potential,

$$G(z,\bar{z}) = \partial_z\Phi = \lim_{\epsilon\to 0}\lim_{N\to\infty}\frac{1}{N}\left\langle \mathrm{Tr}\frac{\bar{z}-X^\dagger}{|z-X|^2+\epsilon^2}\right\rangle. \tag{22}$$

The electric field plays the role of the correct Green's function. Indeed, applying the Gauss law, in the next step,

$$\rho = \frac{1}{\pi}\partial_{\bar{z}}G = \frac{1}{\pi}\partial_{z\bar{z}}\Phi = \lim_{\epsilon\to 0}\lim_{N\to\infty}\frac{1}{\pi N}\left\langle \mathrm{Tr}\frac{\epsilon^2}{[|z-X|^2+\epsilon^2]^2}\right\rangle, \tag{23}$$

we recover the spectral density $\rho(z,\bar{z}) = \frac{1}{N}\langle\sum_{i=1}^N \delta^{(2)}(z-\lambda_i)\rangle$, using the known representation of the two-dimensional delta function $\delta^{(2)}(z) = \lim_{\epsilon\to 0}\frac{1}{\pi}\frac{\epsilon^2}{[|z|^2+\epsilon^2]^2}$. Note that the Gauss law implies the non-analyticity of $G(z,\bar{z})$. It is crucial that the limit $N\to\infty$ is taken first, before taking the infinitesimal regulator ε to zero, since only such order provides the necessary coupling between X and $X^\dagger$, reflected in non-vanishing mixed moments. If one took the limits in an opposite order, X and $X^\dagger$ would decouple, and we would obtain a trivial result $G(z) = 1/z$. The bad news, however, is that the Green's function $G(z,\bar{z})$ (22) is given by a very complicated expression, without any similarity to the standard form of the resolvent.

One may bypass the difficulty by relying on the the algebraic construction for the so-called generalized Green's functions proposed some time ago [19, 20]. First, we notice that

$$\mathrm{Tr}\ln[|z-X|^2+\epsilon^2] = \ln\det[|z-X|^2+\epsilon^2] = \ln\det\begin{pmatrix} z-X & \mathrm{i}\epsilon \\ \mathrm{i}\epsilon & \bar{z}-X^\dagger\end{pmatrix}, \tag{24}$$

where the argument of the last determinant is a $2N\times 2N$ matrix, built out of four $N\times N$ blocks. Let us now define a new operation called a block-trace, defined as $\mathrm{bTr}\equiv\mathbf{1}_2\otimes\mathrm{Tr}_{N\times N}$, which acts in the following way:

$$\mathrm{bTr}\begin{pmatrix} A & B\\ C & D\end{pmatrix}_{2N\times 2N} \equiv \begin{pmatrix}\mathrm{Tr}A & \mathrm{Tr}B\\ \mathrm{Tr}C & \mathrm{Tr}D\end{pmatrix}_{2\times 2}, \tag{25}$$

converting a $2N\times 2N$ block matrix into a 2×2 matrix built out of ordinary traces. Additionally, we define another pair of block matrices

$$Q = \begin{pmatrix} z & -\bar{w}\\ w & \bar{z}\end{pmatrix}, \qquad \mathcal{X} = \begin{pmatrix} X & 0\\ 0 & X^\dagger\end{pmatrix}. \tag{26}$$

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We are now ready to propose the construction of the generalized resolvent (2×2 matrix)

$$\mathcal{G}(z,w) \equiv \begin{pmatrix} \mathcal{G}_{11} & \mathcal{G}_{1\bar{1}} \\ \mathcal{G}_{\bar{1}1} & \mathcal{G}_{\bar{1}\bar{1}} \end{pmatrix} = \frac{1}{N} \left\langle \mathrm{bTr} \frac{1}{Q - \mathcal{X}} \right\rangle, \tag{27}$$

By construction, $\mathcal{G}_{11}$ is equal to the non-analytic resolvent $G(z,\bar{z})$ (22), provided we identify $|w|^2 = \epsilon^2$. Note that the duplication trick allowed us to linearize the problem, since the form of the generalized resolvent (27) has formally the form of the standard resolvent for hermitian matrices. One may ask the question, what role is played by the three remaining elements of the matrix $\mathcal{G}$? Let us recall that the general (non-normal) matrix X is determined in terms of its eigenvalues (Z) and a set of left ($\langle L|$) and right ($|R\rangle$) eigenvectors ($X = \sum_i z_i |R_i\rangle\langle L_i|$), which are bi-orthogonal $\langle L_i|R_j\rangle = \delta_{ij}$. By applying a transformation $S = \mathrm{diag}(R, L)$, $S^{-1} = \mathrm{diag}(L^\dagger, R^\dagger)$ (where $L, R, L^\dagger, R^\dagger$ are $N \times N$ matrices built from the corresponding eigenvectors), we notice that

$$\det(Q - \mathcal{X}) = \det[S^{-1}(Q - \mathcal{X})S] = \det \begin{pmatrix} z - Z & -\bar{w} L^\dagger L \\ w R^\dagger R & \bar{z} - Z^\dagger \end{pmatrix}, \tag{28}$$

so the off-diagonal elements of the generalized Green's functions are related to the expectation values of the overlaps of eigenvectors. Indeed, the left-right eigenvector correlator [22] reads:

$$O(z,\tau) \equiv \frac{1}{N^2} \left\langle \sum_a O_{aa} \delta^{(2)}(z - z_a) \right\rangle, \tag{29}$$

where $O_{ij} = \langle L_i|L_j\rangle\langle R_j|R_i\rangle$ is given in the large N limit by the product of off-diagonal elements of $\mathcal{G}$:

$$\lim_{N\to\infty} O(z,\tau) = -\frac{1}{\pi} \mathcal{G}_{1\bar{1}} \mathcal{G}_{\bar{1}1}|_{w=0}. \tag{30}$$

as was proven in [23]. The appearance of this correlator is a genuine feature of non-hermitian random matrix models, since in the hermitian case left and right eigenvectors coincide and so $O_{ij} = \delta_{ij}$. Finally, for completeness we notice that $\mathcal{G}_{\bar{1}\bar{1}}$ is a complex conjugate of $\mathcal{G}_{11}$ and does not bring any new information.

We would now like to comment on the role of the w variable. In the hermitian case, the method of the resolvent involves the whole complex plane z, despite the fact that the real spectrum comes only as a discontinuity near $z = \lambda \pm \mathrm{i}\epsilon$, corresponding to the imaginary part of the resolvent. In the non-hermitian case, the spectrum is complex, but one may be tempted to probe the generalized Green's function with the complex plane w 'orthogonal' to the plane z, as schematically depicted on figure 1. This choice of strategy is reinforced by the above observed coupling of the w plane to eigenvector correlators.

The promotion of the original regulator $i\epsilon$ to a complex variable w has additional advantages. Firstly, from the algebraic point of view, Q is a quaternion, since $Q = q_0 + \mathrm{i}\sigma_j q_j$, where σ_j are Pauli matrices, so $z = q_0 + \mathrm{i}q_3$ and $w = -q_2 + \mathrm{i}q_1$. This fact significantly simplifies the algebraic calculations, since block matrices such as $\mathcal{X}$ and arguments Q naturally appear in non-hermitian random matrix models, e.g. in the

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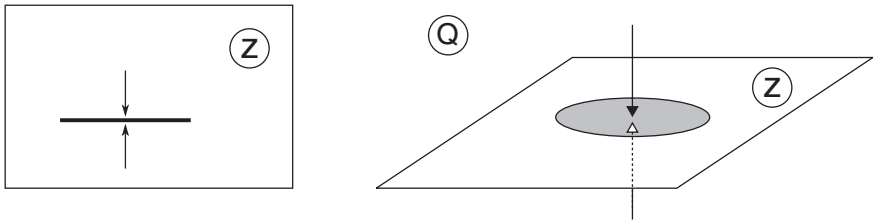

Figure 1. Schematic comparison between the domains of hermitian Green's function $G(z)$ and the non-hermitian, generalized Green's function $\mathcal{G}(Q)$. Arrows on the left figure signal the discontinuity of the Green's function when approaching the cut (solid line), arrows on the right figure denote an additional variable w, which in standard approach is treated as only an infinitesimal regulator. Shaded disc represents the non-analytic domain where the eigenvalues condense.

generalized Green's function technique [19, 20],in hermitization methods [24–26] or in the derivation of the multiplication law for non-hermitian random matrices [27]. The above construction was also recently proven rigorously in the mathematical literature [28]. Secondly, introducing the variable w provides a hint on which object should play the role of the average characteristic polynomial in the case of the GE subjected to Ornstein–Uhlenbeck process, as we now demonstrate.

We define now a determinant expressed in terms of the quaternionic variable Q by

$$D \equiv D(|w|, z, \bar{z}, \tau) \equiv \langle \det(Q - \mathcal{X}) \rangle_\tau = \left\langle \det \begin{pmatrix} z - X & -\bar{w} \\ w & \bar{z} - X^\dagger \end{pmatrix} \right\rangle_\tau , \tag{31}$$

where the measure over which the averaging is performed reflects the Ornstein–Uhlenbeck process. More concretely, it is given by the following averaged increments (compare with the hermitian counterpart (3)):

$$\left\langle \delta X_{ij} \right\rangle = -aX_{ij}\delta\tau, \qquad \left\langle \left| \delta X_{ij} \right|^2 \right\rangle = \frac{1}{N}\delta\tau, \tag{32}$$

which is also expressible as an SFP equation for the jpdf $P(x, y, \tau) = \prod_{i,j=1}^{N} P(x_{ij}, \tau)P(y_{ij}, \tau)$:

$$\partial_\tau P(x, y, \tau) = \mathcal{B}(x, y)P(x, y, \tau), \tag{33}$$

with

$$\mathcal{B} = \frac{1}{4N} \sum_{i,j=1}^{N} (\partial^2_{x_{ij}} + \partial^2_{y_{ij}}) + a \sum_{i,j=1}^{N} (\partial_{x_{ij}} x_{ij} + \partial_{y_{ij}} y_{ij}). \tag{34}$$

Up to an irrelevant normalization constant C, a source-like solution to (33) reads

$$P(X, \tau) = C \exp\left(-\frac{2Na}{1 - \mathrm{e}^{-2a\tau}} \mathrm{Tr} |X - X_0 \mathrm{e}^{-a\tau}|^2 \right). \tag{35}$$

Following similar steps as in the hermitian case, we express the determinant with the help of the auxiliary Grassmann variables and use the properties of the diffusion

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process to arrive (see Appendix B for the details) at the exact (for any matrix size N and for any initial conditions) equation

$$\partial_\tau D = \frac{1}{N}\partial_{w\bar{w}} D - 2NaD + a\mathrm{d}D, \tag{36}$$

with the operator $\mathrm{d} = z\partial_z + \bar{z}\partial_{\bar{z}} + w\partial_w + \bar{w}\partial_{\bar{w}}$.

It is worthy to disentangle the mixed variables present in the last, 'drift' term, by repeating the Lamperti transformation defined by (18) in the hermitian case (with N replaced by $2N$ and Q instead of z). This change of variables leads to the free diffusion

$$\partial_{\tau'} D' = \frac{1}{N}\partial_{w'\bar{w}'} D'. \tag{37}$$

We contrast this equation to its hermitian counterpart (19). Again, it is exactly integrable, and the case of free diffusion was considered by us in [29, 30]. Note that this time the diffusion is two-dimensional, and the Laplace operator acts in the w' space, which, in standard treatments, is largely ignored by serving merely as a regulator. Alike in the hermitian case, we may suspect the emergence of the Burgers structure, provided we apply Cole–Hopf transformation. Since we have at our disposal two complex variables z' and w', we may perform two independent Cole–Hopf transformations

$$g' \equiv \frac{1}{N}\partial_{z'} \ln D', \quad u' \equiv \frac{1}{N}\partial_{w'} \ln D', \tag{38}$$

which satisfy [29, 30]

$$\partial_{\tau'} g' = \frac{1}{N}\partial_{w'\bar{w}'} g' + \partial_{z'}(u'\bar{u'}), \tag{39}$$

$$\partial_{\tau'} u' = \frac{1}{N}\partial_{w'\bar{w}'} u' + \partial_{w'}(u'\bar{u'}). \tag{40}$$

Let us then perform the macroscopic and microscopic limit of the above equations.

3.1. Macroscopic limit

In the large N limit, the second equation (40) takes the form of an inviscid Burgers equation in $2+1$ dimensions

$$\partial_{\tau'} u' = \partial_{w'} |u'|^2. \tag{41}$$

In our case this equation can be simplified due to the rotational symmetry. From now on, we follow the solution presented in [29], modulo the primed variables reflecting the Ornstein–Uhlenbeck process. Introducing $v' = |u'|$ and the radial variable $r' = |w'|$ we recover the Euler equation known from hydrodynamics. The standard solution, using the method of characteristics, yields $v' = v'_0(r' + \tau' v')$, where the initial condition $X_0 = 0$

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corresponds to $v_0'(r') = r'/(|z'|^2 + r'^2)$. We identify the v'^2 with the eigenvector correlator $O(z', \tau')$, with an explicit solution

$$O(z', \tau') = \frac{1}{\pi \tau'^2}(\tau' - |z'|^2). \tag{42}$$

Having the solution for v'^2, we can turn back to the first equation $\partial_{\tau'} g' = \partial_{z'} v'^2$. Elementary integration and initial conditions lead to $g' = \bar{z}'/\tau'$ which in turn gives the spectral density

$$\rho(z', \tau') = \frac{1}{\pi \tau'}\theta(\tau' - |z'|^2). \tag{43}$$

We can now return to the unprimed variables, using the same Lamperti formulae (18) and perform the stationary limit $\tau \to \infty$. In the end, the eigenvector correlator and the spectral density read simply

$$O(z, \bar{z}) = \frac{4a^2}{\pi}\left(\frac{1}{2a} - |z|^2\right), \tag{44}$$

$$\rho(z, \bar{z}) = \frac{2a}{\pi}\theta\left(\frac{1}{2a} - |z|^2\right), \tag{45}$$

which reproduce the known GE results for $a = 1/2$.

It is amusing to note, that historically, the first equation for $O(z, \bar{z})$ in the GE was delivered by Chalker and Mehlig [22] more than three decades after the result for the uniform spectral density $\rho(z, \bar{z})$, originally obtained by Ginibre. In our 'turbulent' formulation, at least in the large N limit, the equation for the eigenvector correlator is of primary importance, and the solution for the spectral density follows trivially from the knowledge of the eigenvector correlator. This observation points at the crucial difference between the hermitian and non-hermitian random matrix models—whereas in the hermitian case the spectral properties are dominant and the eigenvectors decouple, in the non-hermitian case the eigenvectors control the spectral evolution. Technically, this observation was missed in the literature because the analytic structure driven by the w variable was overlooked.

3.2. Microscopic limit

We refer the detailed discussion to the published work [30]. We mention only, that since the equation for characteristic determinant is given as exactly integrable, $2 + 1$ dimensional diffusion equation, valid for any N and any initial condition, unraveling the universal behavior at the edge of the spectrum is a consequence of certain limiting procedures of the exact solution. Interestingly, the pre-shock wave appears in the non-hermitian case in the eigenvector correlator, contrary to the appearance of the spectral pre-shock wave in eigenvalue spectrum in the case of the hermitian ensemble. Looking at the neighborhood of the shock by parameterizing the fluctuations in the vicinity of

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the boundary as $|z|-1=\eta N^{-1/2}$, we recover the known universal result for the microscopic behavior

$$\rho(\eta)\approx\frac{1}{2\pi}\mathrm{Erfc}(\sqrt{2}\,\eta). \tag{46}$$

The form of the unfolding is expected from the general geometric argument since the number of eigenvalues on the surface of the disc grows alike N, the scaling on the boundary has to grow alike $\sqrt{N}$. Note that contrary to the hermitian case, the spectrum does not oscillate wildly at the edge, but rather smoothly interpolates between the plateau at $1/\pi$ and 0. This behavior can be linked to the fact, that the viscosity in the non-hermitian case has a positive sign.

4. A qualitative relationship between the dynamics of eigenvectors and eigenvalues based on numerical experiments

To gain some more insight into the intertwined dynamics of the complex eigenvalues and eigenvectors of non-hermitian matrices, let us perform some numerical experiments. First, for comparison, we turn to hermitian matrices. As was mentioned before, in this case, the diffusion (4) induces the Langevin equation (1) for the eigenvalues—no coupling to the eigenvectors is present. It is worth mentioning that the eigenvector dynamics *does* depend on the eigenvalues. The associated stochastic equation of motion is given by:

$$\mathrm{d}|\psi_i(\tau)\rangle=\frac{1}{\sqrt{N}}\sum_{j=1,j\neq i}^{N}\frac{\mathrm{d}B_{ij}(\tau)}{\lambda_i-\lambda_j}|\psi_j\rangle-\frac{1}{2N}\sum_{j=1,j\neq i}^{N}\frac{\mathrm{d}\tau}{(\lambda_i-\lambda_j)^2}|\psi_i\rangle, \tag{47}$$

and has been studied extensively in [34] (the $|\psi_i\rangle$ is an eigenvector corresponding to eigenvalue λ_i, $\mathrm{d}B_{ij}$ is a multi-dimensional Brownian motion). Nonetheless, the evolution of the eigenvalues does not depend on the eigenvectors and when interested only in the dynamics of λ_i, we can ignore the changes of $|\psi_i\rangle$s. The resulting process is depicted in figure 2 where we present the eigenvalue trajectories of a $N=20$ hermitian matrix initiated with two distinct eigenvalues $\lambda=-1,1$ with equal multiplicities. Additionally, we have computed the jump amplitude of a particular eigenvalue normalized by the simulations time step. Note that there is no distinct dependence of the jump on how close the eigenvalue is to its neighbours.

In the case of non-hermitian dynamics, the Langevin equations are not readily available—we relegate their derivation and study to future work. Nonetheless, as was argued in section 3, we now know that the eigenvectors and in particular O_{ii}s are crucial for the dynamics of complex eigenvalues. To show this, we focus on an example of a non-hermitian evolution of a $N=2$ matrix starting from $\mathrm{diag}(-0.3,0.3)$. In figure 3, we observe the eigenvalues covering the complex plane in a diffusive manner. It is also expected that they repel each other. To perform a closer inspection (see figure 4), we plot three characteristics of their dynamics—the distance between the eigenvalues $|\lambda_1-\lambda_2|$, the eigenvector correlator $O_{11}=\langle L_1|L_1\rangle\langle R_1|R_1\rangle$ and the normalized jump

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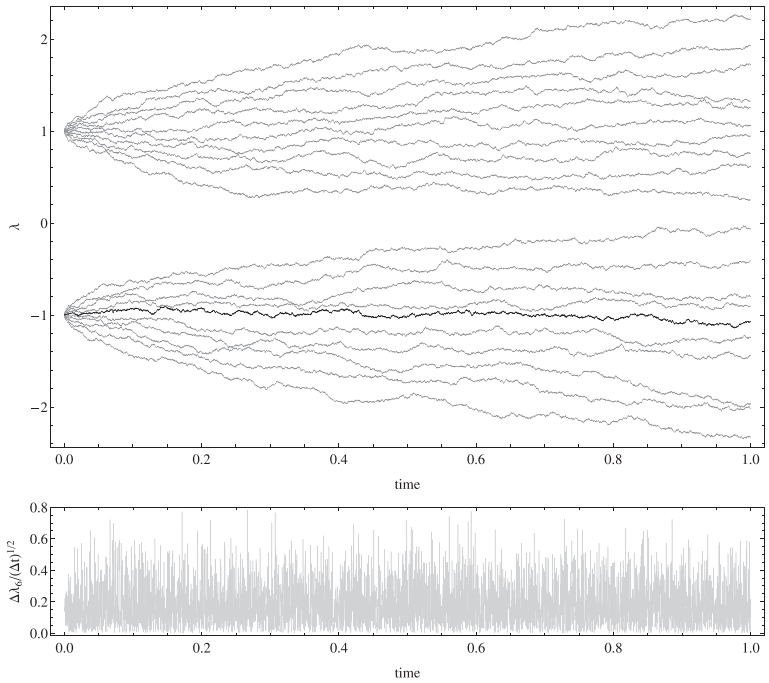

Figure 2. Upper inlet, a single numerical realization of stochastic behaviour of $N = 20$ eigenvalues governed by equation (1). Initially the eigenvalues were put at -1, 1 with equal multiplicities. Lower inlet, the jump amplitude of a given eigenvalue (in bold in the upper plot), normalized by the square root of the time step used in the simulation.

$\Delta\lambda_1/(\Delta t)^{1/2}$ of the first eigenvalue, all as a function of time. We chose to ignore both $O_{22} = O_{11}$ and $O_{12} = 1 - O_{11}$ since they do not offer any additional information. The most interesting feature of this particular realization occurs around the time $t_c = 0.1$ of minimal eigenvalue distance (this precise moment is depicted by white dots on figure 4). We observe that as the distance gets smaller, the O_{11} blows up in a correlated manner. This is accompanied by an increase in the jump amplitude of the eigenvalue. We have checked that this effect prevails when matrix size is larger than two. Note again that it was not present for eigenvalues of hermitian matrices, for which the distance between the eigenvalues also drives the evolution. We therefore consider this effect as a qualitative demonstration of the co-dependence between the evolutions of eigenvalues and eigenvectors in this scenario.

5. Unexpected links

Several unexpected links between the static hermitian and non-hermitian random matrix models were noted in the past [20]. The spectrum of hermitian matrices is real, but the main tool relies on introducing the complex valued resolvent (Green's function), whose discontinuities allow to infer the spectral function, using the theory of analytic

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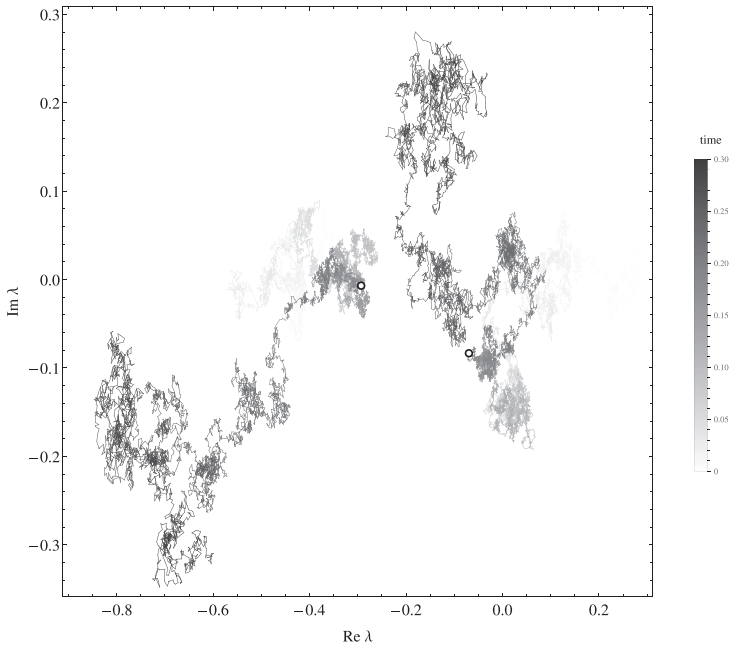

Figure 3. Numerical realization of a stochastic behaviour of $N = 2$ eigenvalues of a non-hermitian matrix diffusing according to (32), with $a = 0$. Initial conditions are $\lambda_1 = 0.3, \lambda_2 = -0.3$ and the color of paths encode time evolution. Black edged white dots represent the position of λs at time $t = 0.1$ when the distance is minimized (see figure 4).

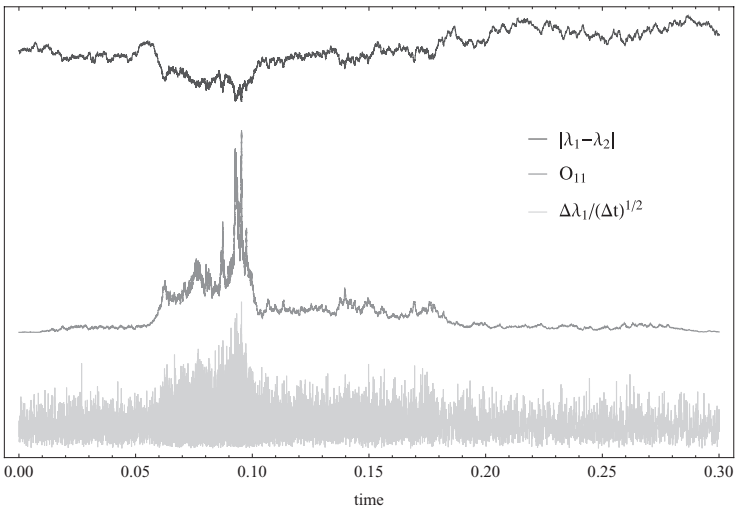

Figure 4. Time series of eigenvalue distance $|\lambda_1 - \lambda_2|$, eigenvector O_{11} and eigenvalue diffusion distance $\delta\lambda_1 = \Delta\lambda_1/(\Delta t)^{1/2}$. Corresponding vertical axes are out of scale, we identify the time $t_c = 0.1$ to be of both minimal distance $|\lambda_1 - \lambda_2|$ and maximal values of O_{11} and $\delta\lambda_1$.

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functions. In the large N limit, a particular transform, known as the R-transform, related to the Green's function by the functional inverse as $R[G(z)] + 1/G(z) = z$, plays the role of the analog of generating function of classical cumulants in the matrix-valued probability calculus. The R-transform constitutes the cornerstone of the free probability theory [31] and generates matrix-valued analogues of classical central limit theorems. In the case of non-hermitian matrices, the spectrum is complex, but the regulator ϵ^2 in the logarithmic potential (21) behaves as the tip of an iceberg, pointing at a hidden algebraic structure. Indeed, in order to maintain the analogy to the hermitian case, one has to embed the structure of the generalized Green's functions in the algebra of quaternions. In such a way, a second complex variable w, 'perpendicular' to z emerges. In the large N limit, one can adapt the Voiculescu construction for the R-transform by defining the quaternion valued functional inverse $\mathcal{R}[\mathcal{G}(Q)] + 1/\mathcal{G}(Q) = Q$ and thus allowing for non-hermitian and non-commuting convolution of random matrices [19, 20]. Surprisingly, the links between the hermitian and non-hermitian random matrix models stretch out to the area of dynamic processes. In the case of the Gaussian randomness, the exact diffusion equation for the averaged characteristic polynomial finds its exact analogue for the averaged characteristic polynomial valued in the algebra of quaternions. It turns out, that the 'hidden' variable w, ignored in standard treatment of non-hermitian random matrix models, plays a crucial role in determining the two-dimensional pattern of the spectral evolution. In the large N limit, hermitian and non-hermitian SFP equations take the surprisingly similar form of a Burgers-like structure. In general, the Voiculescu equation $\partial_\tau G + R(G)\partial_z G = 0$ is replaced by its quaternionic counterpart [32]

$$\frac{\partial \mathcal{G}_{ab}}{\partial \tau} + \sum_{c,d=1}^{2} \mathcal{R}[\mathcal{G}]_{cd} \frac{\partial \mathcal{G}_{ab}}{\partial Q_{cd}} = 0, \tag{48}$$

where Latin indices label the two-by two quaternionic structure of Q, $\mathcal{G}$ and $\mathcal{R}$. In both cases, singularities emerge. However, in the hermitian case, singularities appear in the flow of the eigenvalues, whereas in the case of non-hermitian ensembles, singularities appear in the flow of a certain correlator of left and right eigenvectors. In both cases, finite N effects can be taken into account as an appearance of spectral viscosity proportional to $1/N$. There is however a crucial difference in the sign—positive spectral viscosity smoothens the edge of the Ginibre spectrum, yielding universal behavior given by the Erfc function, whereas negative spectral viscosity in the GUE triggers violent oscillations, leading to the formation of the so-called Airy kernel. Resolving the deep reasons for these links still remains one of the challenges of random matrix models. We summarize the unexpected links between the GUE and GE ensembles in table 1.

6. Conclusions

The presented results borrow to a large extent from the conclusions obtained in the series of papers of the present authors [12, 18, 29, 30], but also include new solutions. First, we adapted the turbulent scenario to the Ornstein–Uhlenbeck process for GUE. Technical details are deferred to appendix A. Then, by a set of Lamperti

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Table 1. Comparison of links between GUE and GE.

	GUE	GE
Spectral density	Real	Complex
Resolvent	Complex-valued $G(z) = \frac{1}{N}\langle \mathrm{Tr}(z-H)^{-1}\rangle$	Quaternion-valued $\mathcal{G}(Q) = \frac{1}{N}\langle \mathrm{bTr}(Q-\mathcal{X})^{-1}\rangle$
Determinant	$U(z,\tau) = \langle \det(z-H)\rangle$	$D(Q,\tau) = \langle \det(Q-\mathcal{X})\rangle$
Diffusion equation	$\partial_\tau U = -\frac{1}{2N}\partial_{zz} U$	$\partial_\tau D = +\frac{1}{N}\partial_{w\bar{w}} D$
Viscosity	Negative	Positive
Universal behavior	Oscillatory (Airy kernel)	Smooth (Erfc)
R-transform	$R_{\mathrm{GUE}}(G) = G$	$\mathcal{R}_{GG}(\mathcal{G}) = \begin{pmatrix} 0 & \mathcal{G}_{1\bar{1}} \\ \mathcal{G}_{\bar{1}1} & 0 \end{pmatrix}$
Voiculescu equation	$\frac{\partial G}{\partial \tau} + R(G)\frac{\partial G}{\partial z} = 0$	$\frac{\partial \mathcal{G}_{ab}}{\partial \tau} + \sum_{c,d=1}^{2} \mathcal{R}[\mathcal{G}]_{cd}\frac{\partial \mathcal{G}_{ab}}{\partial Q_{cd}} = 0$
Pre-shock waves	Flow of eigenvalues	Flow of eigenvector correlators

transformations, we provided an exact mapping between the Ornstein–Uhlenbeck process and free diffusion. This mapping allowed us to interpolate smoothly between the microscopic limit (Dyson's local equilibrium) and the macroscopic limit (Dyson's global equilibrium). Second, we have repeated the same scenario of the Ornstein–Uhlenbeck process for the GE. Again, we relegate technical details to appendix B. Last but not least, we tried to point at rather unexpected analogies and similarities in both examples. We stressed that such analogies are detectable only when the quaternion variables are used.

We have proven that a consistent description of non-hermitian ensembles require the knowledge of the detailed dynamics of co-evolving eigenvalues and eigenvectors. Moreover, at least in the large N limit, the dynamics of eigenvectors plays a major role and leads directly to the inference of the spectral properties. This is a dramatically different scenario comparing to the standard random matrix models, where the statistical properties of eigenvalues are of primary importance, and the properties of eigenvectors are basically trivial due to the their decoupling from the spectra and the fact that they are Haar distributed on $U(N)$. We conjecture that the hidden dynamics of eigenvectors observed in the GE, is a general feature of all non-hermitian random matrix models.

Our formalism could be exploited to expand the area of application of non-hermitian random matrix ensembles within problems of growth, charged droplets in quantum Hall effect and gauge theory/geometry relations in string theory beyond the subclass of complex matrices represented by normal matrices.

One of the challenges is an explanation as to, why, despite being so different, the SFP equations for hermitian and non-hermitian random matrix models exhibit structural similarity to simple models of turbulence, where the so-called Burgers equation plays the vital role, establishing the flow of the spectral density of eigenvalues in the case of the hermitian or unitary ensembles and the flow of certain eigenvector

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correlator in the case of non-hermitian ensembles. Another challenge surrounds in completing the Langevin-like equations (1) adapted for non-hermitian cases.

We believe that our findings will contribute to the understanding of several puzzles of non-hermitian dynamics, alike extreme sensitivity of spectra of non-hermitian systems to perturbations [22, 33] and the sign problem of certain Euclidean Dirac operators.

Acknowledgments

MAN thanks the organizers of the 7th International Conference on Unsolved Problems on Noise, Barcelona, Spain, for hospitality. We appreciate collaboration with Z Burda, E Gudowska-Nowak, R Janik, A Jarosz, J Jurkiewicz, G Papp and I Zahed and discussions with N O'Connell, Y Fyodorov, M Kieburg, R Tribe, D Savin and O Zaboronski. This work was supported by the Grant DEC-2011/02/A/ST1/00119 of the National Centre of Science. WT appreciates partial support from the Ministry of Science and Higher Education through the Grant 0225/DIA/2015/44.

Appendix A

In what follows we derive the partial differential equation (10). For completeness, we again introduce the Ornstein–Uhlenbeck diffusion process in terms of an SFP equation (6):

$$\partial_\tau P(x,y,\tau) = \mathcal{A}(x,y)P(x,y,\tau), \tag{A.1}$$

with the Laplace operator

$$\mathcal{A}(x,y) = \sum_{k=1}^{N}\left(\frac{1}{2N}\frac{\partial^2}{\partial x_{kk}^2} + a\frac{\partial}{\partial x_{kk}}x_{kk}\right) + \frac{1}{4N}\sum_{i<j=1}^{N}\left(\frac{\partial^2}{\partial x_{ij}^2} + \frac{\partial^2}{\partial y_{ij}^2}\right) + a\sum_{i<j=1}^{N}\left(\frac{\partial}{\partial x_{ij}}x_{ij} + \frac{\partial}{\partial y_{ij}}y_{ij}\right). \tag{A.2}$$

As a first step, we write the determinant as a Gaussian integral over a set of Grassmann variables $\eta_i, \bar{\eta}_i$:

$$\det A = C\int\prod_{i,j=1}^{N} \mathrm{d}\eta_i \mathrm{d}\bar{\eta}_j \exp(\overline{\eta}_i A_{ij}\eta_j). \tag{A.3}$$

with an irrelevant proportionality constant C. This allows us to express the characteristic polynomial U defined by (9) in the following way:

$$U(z,t) = C\int\mathcal{D}[\bar{\eta},\eta,x,y]P(x,y,\tau)\exp\left[\sum_{i,j=1}^{N}\bar{\eta}_i(z\delta_{ij} - H_{ij})\eta_j\right], \tag{A.4}$$

where the integration measure is defined by

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$$\mathcal{D}[\bar{\eta},\eta,x,y] \equiv \prod_{i,j=1}^{N} \mathrm{d}\eta_i \mathrm{d}\bar{\eta}_j \prod_{k=1}^{N} \mathrm{d}x_{kk} \prod_{n<m=1}^{N} \mathrm{d}x_{nm}\mathrm{d}y_{nm}. \tag{A.5}$$

The hermiticity condition ($H_{ij} = \bar{H}_{ji}$) is used to write the argument of the exponent of (A.4) in an explicit form:

$$\begin{aligned} T_g(\bar{\eta},\eta,x,y,z) &\equiv \sum_{i,j=1}^{N} \bar{\eta}_i(z\delta_{ij} - H_{ij})\eta_j = \sum_{r=1}^{N} \bar{\eta}_r(z - x_{rr})\eta_r \\ &+ - \sum_{n<m=1}^{N} \left[x_{nm}(\bar{\eta}_n\eta_m + \bar{\eta}_m\eta_n) + iy_{nm}(\bar{\eta}_n\eta_m - \bar{\eta}_m\eta_n)\right]. \end{aligned} \tag{A.6}$$

Now we make use of the diffusion equation (A.1) which, after integrating by parts, gives:

$$\partial_\tau U = \int \mathcal{D}\partial_\tau P \exp(T_g) = \int \mathcal{D}\mathcal{A}P \exp(T_g) = \int \mathcal{D}P\tilde{\mathcal{A}} \exp(T_g), \tag{A.7}$$

with $\tilde{\mathcal{A}} = \mathcal{A}(a \to -a)$. We calculate the expression:

$$\begin{aligned} \tilde{\mathcal{A}} \exp(T_g) = &\left[a \sum_{k=1}^{N} x_{kk}\bar{\eta}_k\eta_k + \frac{1}{N} \sum_{i<j=1}^{N} \bar{\eta}_i\eta_j\bar{\eta}_j\eta_i\right] \exp(T_g) \\ &+ \left[a \sum_{i<j=1}^{N} [(x_{ij} + \mathrm{i}y_{ij})\bar{\eta}_i\eta_j + (x_{ij} - \mathrm{i}y_{ij})\bar{\eta}_j\eta_i]\right] \exp(T_g). \end{aligned} \tag{A.8}$$

by schematically writing down the action of derivatives on $\exp(T_g)$:

$$\partial_{x_{kk}} \quad \to \quad -\bar{\eta}_k\eta_k, \qquad \partial^2_{x_{kk}} \quad \to \quad 0,$$

and for $i \neq j$:

$$\begin{aligned} \partial_{x_{ij}} &\to -(\bar{\eta}_i\eta_j + \bar{\eta}_j\eta_i), \\ \partial_{y_{ij}} &\to -\mathrm{i}(\bar{\eta}_i\eta_j - \bar{\eta}_j\eta_i), \\ \partial^2_{x_{ij}} &\to (\bar{\eta}_i\eta_j + \bar{\eta}_j\eta_i)(\bar{\eta}_i\eta_j + \bar{\eta}_j\eta_i) = -2\bar{\eta}_i\eta_j\bar{\eta}_j\eta_i, \\ \partial^2_{y_{ij}} &\to -(\bar{\eta}_i\eta_j - \bar{\eta}_j\eta_i)(\bar{\eta}_i\eta_j - \bar{\eta}_j\eta_i) = -2\bar{\eta}_i\eta_j\bar{\eta}_j\eta_i. \end{aligned}$$

We rewrite the terms of (A.8) accordingly:

$$\sum_{i=1}^{N} \bar{\eta}_i\eta_i \exp(T_g) = \partial_z \exp(T_g),$$

$$\sum_{i<j=1}^{N} \bar{\eta}_i\eta_i\bar{\eta}_j\eta_j \exp(T_g) = \frac{1}{2}\partial_{zz} \exp(T_g),$$

$$\sum_{i=1}^{N} \bar{\eta}_i\partial_{\bar{\eta}_i} \exp(T_g) = \left[z\partial_z - \sum_{i=1}^{N} x_{ii}\bar{\eta}_i\eta_i - \sum_{i<j=1}^{N} [(x_{ij} + \mathrm{i}y_{ij})\bar{\eta}_i\eta_j + (x_{ij} - \mathrm{i}y_{ij})\bar{\eta}_j\eta_i]\right] \exp(T_g).$$

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We obtain thus, after recalling (A.7) and (A.8):

$$\partial_\tau U = \int \mathcal{D}P\left(-\frac{1}{2N}\partial_{zz} + az\partial_z - a\sum_{i=1}^{N}\bar{\eta}_i\partial_{\bar{\eta}_i}\right)\exp(T_g), \tag{A.9}$$

where the last term is explicitly calculable upon integrating by parts

$$\int \mathcal{D}P\sum_{i=1}^{N}\bar{\eta}_i\partial_{\bar{\eta}_i}\exp(T_g) = N\int \mathcal{D}P\exp(T_g), \tag{A.10}$$

so that we arrive at equation (10)

$$\partial_\tau U(z,t) = -\frac{1}{2N}\partial_{zz}U(z,\tau) + az\partial_z U(z,\tau) - aNU(z,\tau). \tag{A.11}$$

Appendix B

Here we present the derivation for the evolution equation (36). The Ornstein–Uhlenbeck process in the non-hermitian case is given by an SFP equation for the jpdf $P(x, y, \tau)$:

$$\partial_\tau P(x,y,\tau) = \mathcal{B}(x,y)P(x,y,\tau), \tag{B.1}$$

with the operator

$$\mathcal{B} = \frac{1}{4N}\sum_{i,j=1}^{N}(\partial^2_{x_{ij}} + \partial^2_{y_{ij}}) + a\sum_{i,j=1}^{N}(\partial_{x_{ij}}x_{ij} + \partial_{y_{ij}}y_{ij}). \tag{B.2}$$

To proceed, we open the determinant defined in (31) with the help of Grassmann variables η_i, ξ_i:

$$D = C'\int \mathcal{D}[\ldots]P(x,y,\tau)\exp\left[\begin{pmatrix}\bar{\eta} & \bar{\xi}\end{pmatrix}\begin{pmatrix}z-X & -\bar{w}\\ w & \bar{z}-X^\dagger\end{pmatrix}\begin{pmatrix}\eta\\ \xi\end{pmatrix}\right],$$

with $X_{ij} = x_{ij} + \mathrm{i}y_{ij}$, $X^\dagger_{ij} = x_{ji} - \mathrm{i}y_{ji}$, an irrelevant constant C' and the joint measure

$$\mathcal{D}[\bar{\eta},\eta,\bar{\xi},\xi,x,y] = \prod_{i=1}^{N}\mathrm{d}\eta_i\mathrm{d}\bar{\eta}_i\mathrm{d}\xi_i\mathrm{d}\bar{\xi}_i\prod_{i,j=1}^{N}\mathrm{d}x_{ij}\mathrm{d}y_{ij}. \tag{B.3}$$

The argument of the exponent is equal to

$$\begin{aligned}S_g &\equiv \sum_{i,j=1}^{N}[\bar{\eta}_i(z-X)_{ij}\eta_j + \bar{\xi}_i(\bar{z}-X^\dagger)_{ij}\xi_j + w\bar{\xi}_i\eta_i - \bar{w}\bar{\eta}_i\xi_i]\\ &= \sum_{i,j=1}^{N}[-x_{ij}(\bar{\eta}_i\eta_j + \bar{\xi}_j\xi_i) - \mathrm{i}y_{ij}(\bar{\eta}_i\eta_j - \bar{\xi}_j\xi_i) + z\bar{\eta}_i\eta_i + \bar{z}\bar{\xi}_i\xi_i + w\bar{\xi}_i\eta_i - \bar{w}\bar{\eta}_i\xi_i].\end{aligned}$$

We make use of the diffusion SFP equation and integrate by parts:

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$$\partial_\tau D = \int \mathcal{D}\partial_\tau P \exp(S_g) = \int \mathcal{D}\mathcal{B}P \exp(S_g) = \int \mathcal{D}P\tilde{\mathcal{B}} \exp(S_g), \tag{B.4}$$

where $\tilde{\mathcal{B}} = \mathcal{B}(a \to -a)$. The last integrand reads:

$$\tilde{\mathcal{B}} \exp(S_g) = \left[\frac{1}{N} \sum_{i,j=1}^{N} \bar{\eta}_i \eta_j \bar{\xi}_j \xi_i + a \sum_{i,j=1}^{N} ((x_{ij} + \mathrm{i}y_{ij}) \bar{\eta}_i \eta_j + (x_{ij} - \mathrm{i}y_{ij}) \bar{\xi}_j \xi_i) \right] \exp(S_g), \tag{B.5}$$

where we used the schematic formulas acting on $\exp(S_g)$:

$$\begin{aligned}
\partial_{x_{ij}} &\to -(\bar{\eta}_i \eta_j + \bar{\xi}_j \xi_i), \\
\partial_{y_{ij}} &\to -\mathrm{i}(\bar{\eta}_i \eta_j - \bar{\xi}_j \xi_i), \\
\partial^2_{x_{ij}} &\to (\bar{\eta}_i \eta_j + \bar{\xi}_j \xi_i)(\bar{\eta}_i \eta_j + \bar{\xi}_j \xi_i) = 2\bar{\eta}_i \eta_j \bar{\xi}_j \xi_i, \\
\partial^2_{y_{ij}} &\to -(\bar{\eta}_i \eta_j - \bar{\xi}_j \xi_i)(\bar{\eta}_i \eta_j - \bar{\xi}_j \xi_i) = 2\bar{\eta}_i \eta_j \bar{\xi}_j \xi_i.
\end{aligned}$$

To continue, we rewrite the terms of (B.5) proportional to a as follows:

$$\begin{aligned}
&\sum_{i=1}^{N} \bar{\eta}_i \eta_i \exp(S_g) = \partial_z \exp(S_g), \qquad \sum_{i=1}^{N} \bar{\xi}_i \xi_i \exp(S_g) = \partial_{\bar{z}} \exp(S_g), \\
&\sum_{i=1}^{N} \bar{\eta}_i \xi_i \exp(S_g) = -\partial_{\bar{w}} \exp(S_g), \qquad \sum_{i=1}^{N} \bar{\xi}_i \eta_i \exp(S_g) = \partial_w \exp(S_g), \\
&\sum_{i=1}^{N} \bar{\eta}_i \partial_{\bar{\eta}_i} \exp(S_g) = \left(- \sum_{i,j=1}^{N} (x_{ij} + \mathrm{i}y_{ij}) \bar{\eta}_i \eta_j + z \sum_{i=1}^{N} \bar{\eta}_i \eta_i - \bar{w} \sum_{i=1}^{N} \bar{\eta}_i \xi_i \right) \exp(S_g), \\
&\sum_{j=1}^{N} \bar{\xi}_j \partial_{\bar{\xi}_j} \exp(S_g) = \left(- \sum_{i,j=1}^{N} (x_{ij} - \mathrm{i}y_{ij}) \bar{\xi}_j \xi_i + \sum_{j=1}^{N} \bar{z} \bar{\xi}_j \xi_j + w \sum_{j=1}^{N} \bar{\xi}_j \eta_j \right) \exp(S_g),
\end{aligned}$$

so that

$$\begin{aligned}
&a \sum_{i,j=1}^{N} ((x_{ij} + \mathrm{i}y_{ij}) \bar{\eta}_i \eta_j + (x_{ij} - \mathrm{i}y_{ij}) \bar{\xi}_j \xi_i) \exp(S_g) \\
&= \left[a \sum_{i=1}^{N} (-\bar{\eta}_i \partial_{\bar{\eta}_i} - \bar{\xi}_j \partial_{\bar{\xi}_j}) + a(z\partial_z + \bar{z}\partial_{\bar{z}} + w\partial_w + \bar{w}\partial_{\bar{w}}) \right] \exp(S_g).
\end{aligned} \tag{B.6}$$

Plugging the above expressions into (B.4), gives:

$$\begin{aligned}
\partial_\tau D = \int \mathcal{D}P \Bigg(&\frac{1}{N} \sum_{i,j=1}^{N} \bar{\eta}_i \eta_j \bar{\xi}_j \xi_i - a \sum_{i=1}^{N} (\bar{\eta}_i \partial_{\bar{\eta}_i} + \bar{\xi}_j \partial_{\bar{\xi}_j}) \\
&+ a(z\partial_z + \bar{z}\partial_{\bar{z}} + w\partial_w + \bar{w}\partial_{\bar{w}}) \Bigg) \exp(S_g),
\end{aligned} \tag{B.7}$$

where the first term is expressible as

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$$\partial_{w\bar{w}} D = \int \mathcal{D}P \sum_{i,j=1}^{N} \bar{\eta}_j \eta_i \bar{\xi}_i \xi_j \exp(S_g), \tag{B.8}$$

and the second one reads

$$\int \mathcal{D}P \sum_{i=1}^{N} (\bar{\eta}_i \partial_{\bar{\eta}_i} + \bar{\xi}_j \partial_{\bar{\xi}_j}) \exp(S_g) = 2N \int \mathcal{D}P \exp(S_g). \tag{B.9}$$

Taking them into account we finally obtain equation (36)

$$\partial_\tau D = \frac{1}{N} \partial_{w\bar{w}} D - 2NaD + a\mathrm{d}D, \tag{B.10}$$

where $\mathrm{d} = z\partial_z + \bar{z}\partial_{\bar{z}} + w\partial_w + \bar{w}\partial_{\bar{w}}$.

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Exact spectral densities of complex noise-plus-structure random matrices

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We use supersymmetry to calculate exact spectral densities for a class of complex random matrix models having the form $M = S + LXR$, where X is a random noise part X and S, L, R are fixed structure parts. This is a certain version of the "external field" random matrix models. We found two–fold integral formulas for arbitrary structural matrices. We investigate some special cases in detail and carry out numerical simulations. The presence or absence of a normality condition on S leads to a qualitatively different behavior of the eigenvalue densities.

arXiv:1605.01159v1 [math-ph] 4 May 2016

I. NOISE-PLUS-STRUCTURE RANDOM MATRICES

In the last 50 years, Random Matrix Theory (RMT) has been established as an impressively versatile approach [4] of studying complex systems. In particular, applications include large data structures [34], machine learning algorithms [1] and telecommunications [14] arose recently. It is a common problem in these and many other areas to infer a signal or information from noisy data. In this work we study a type of RMT noise-plus-structure model suitable for this type of inference tasks. More specifically, let M be a matrix of the form:

$$M = S + LXR, \tag{1}$$

where S is a fixed matrix and $L, R > 0$ are diagonal positive definite covariance matrices. The matrix X is the source of noise drawn typically from a multi-dimensional Gaussian ensemble. Equation (1) thus comprises a simplest model combining both randomness (X) and structure (S, L, R). The matrix S is called a source and is interpreted as the signal/information matrix of the system in study. We add a structured noise LXR as every real–world data is contaminated, and only the resulting matrix M is attainable by experiment. The matrices L, R encode an anisotropic (or correlated) source of randomness — a single element of the source matrix S_{ij} is perturbed by a noisy term $L_{ii}R_{jj}X_{ij}$, *i.e.* with variance $\sigma_{ij}^2 = (L_{ii}R_{jj})^2$. Absence of any structure means setting $S = 0$ and $L = R = 1$ which reduces Eq. (1) to standard RMT models of pure randomness.

There are at least two strategies of studying the model (1) — we look at either the eigenvalues or the singular values of M (equivalently at the eigenvalues of $M^\dagger M$). The first approach is limited to square matrices whereas the second route is the main idea behind the Principal Component Analysis in which, in general, rectangular data matrices M are investigated. In this work we focus on the first approach and study the statistics of the eigenvalues. It is well-known that the symmetries of M constrain the position of its eigenvalues. Here, however, we drop any symmetry constraints and focus on the case where eigenvalues spread over the whole complex plane. In what follows we discuss a couple of instances which can be realized with the model (1) and which are interesting from a practical as well as from a theoretical perspective.

In finance, one studies the markets to make educated guesses of their future behaviour, including the search for possibly profitable correlations. To this end one typically considers N assets in T time slices which may be ordered in a rectangular $N \times T$ matrix M. We set $S = 0$ and interpret L, R as noise correlation matrices in both time and space. Because M is rectangular, the spectral density of $M^T M$ is studied and thus we arrive at the doubly correlated Wishart model [38]. As a second example, in wireless telecommunication Eq. (1) arises in Multiple Input Multiple Output (MIMO) systems as a complex $N_r \times N_t$ transmission matrix M between N_t transmitters and N_r receivers [31].

As a physics application, we consider a Hermitian Hamiltonian M which models an ensemble of charged spinless particles interacting with a strong external magnetic field [29]. In this instance we set $S = e^{-\tau}H_0$, $LR = \sqrt{1 - e^{-2\tau}}$ and both H_0 and X are random matrices drawn from the Gaussian Unitary Ensemble (GUE). The parameter τ is proportional to the applied magnetic field. For moderate fields a different Random Matrix Model of (1) applies — a transition between a Gaussian Orthogonal Ensemble (GOE) and a GUE happens due to the breaking of time reversal invariance. In this regime we set $LR = i\alpha$ while the random matrices S and X are symmetric $S = S^T$ and X antisymmetric $X = -X^T$, respectively. Even though we drop the positivity condition of L, R and consider a random matrix S, the model described is still of the form (1). As the parameter α which is proportional to the field varies between $0 \to 1$, a transition between GOE and GUE takes place.

Independently, the rich mathematical structure of models of the type (1) has attracted a lot of attention in its own right. These ensembles are known in the RMT community as "external source models". So far they were mostly considered for $L = R = 1$ and Hermitian X [10, 11, 16, 21]. These models also have a natural

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interpretation in terms of Dyson's Brownian motion for the stochastic evolution in time τ, when we set $LR = \sqrt{\tau}$ and view S as the initial matrix [8, 9].

Although all of the above examples contain either complex or real matrices M with a purely real spectrum, there are situations where symmetry constraints are not present and the spectrum spreads over the whole complex plane. One of the main tenets of quantum mechanics for closed systems is the Hermiticity of the Hamiltonian, while dropping it is an often used effective way to describe open systems, *i.e.* to account for the environment. As a consequence, complex energies of the type $E = \varepsilon - i\Gamma$ arise which correspond to resonant states. Such an energy eigenstate $|\phi_E(t)\rangle = e^{-iEt}|\phi_E(0)\rangle$ does not only oscillate with a frequency ε but also decays with a characteristic time $1/\Gamma$. Random Matrix Models of this type were used for studying quantum chaotic scattering in open cavities [18]. In this case, the matrix S is drawn from the GUE, $LR = -i\pi$, and $X = W^\dagger W$ models a random interaction between the cavity and its surroundings, where W is drawn from a complex Girko–Ginibre Ensemble.

As a second application of non–Hermitian matrices, we mention efforts in constructing mathematical models of neuronal networks [30, 35]. Here, M represents the neuronal adjacency matrix and we begin with setting $S = 0, L = R = 1$. In this context however, an additional constraint is needed — each matrix row must be either purely negative or purely positive which reflects Dale's Law of neuronal behaviour. Moreover, a recent paper [3] argued that also the S, L and R matrices in the model (1) might be of significance.

In the sequel, we consider matrices X drawn from the Girko–Ginibre Ensemble (*i.e.*, a matrix with complex Gaussians random entries) as well as various types of structural matrices S, L and R. In Sec. II we compute an exact formula for the spectral density of M and arbitrary matrices S, L and R. In Sec. III we investigate particular cases: a normal matrix S and arbitrary matrices L, R, a vanishing source $S = 0$ and trivial $L = R = 1$, and a rank–one non–normal source S with $L = R = 1$. Eventually, we comment on the spectral formula for a related problem of eigenvalues of M^{-1}. We summarize and conclude in Sec. IV.

II. SPECTRAL DENSITY OF M

We now describe the model (1) in greater detail. Let X be an $N \times N$ matrix drawn from a complex Girko–Ginibre Ensemble,

$$P(X)dX = C^{-1} \exp\left(-n \mathrm{Tr} X^\dagger X\right) dX, \tag{2}$$

where n is an (inverse) variance parameter and $C = (\pi/n)^{N^2}$ is the normalization constant. The flat measure over the matrices X is denoted dX. All matrices S, L and R are $N \times N$, with L, R being positive definite and diagonal. The source matrix S is in the most general form given by $S = D + T$ where D is diagonal and T is strictly upper triangular. These reduced forms are not restrictive because the spectrum of M is unitarily invariant. In particular, the Schur decomposition of the source matrix reads $S = U^\dagger(D + T)U$ for a particular unitary matrix U. When $T = 0$ the source matrix is called normal, otherwise it is non-normal.

A basic statistical quantity characterizing the model (1) is the spectral density

$$\rho(z, \bar{z}) = \frac{1}{N} \left\langle \sum_{i=1}^{N} \delta^{(2)}(z - m_i) \right\rangle_P, \tag{3}$$

depending on the complex variable z. The m_i are the eigenvalues of M. We use the two–dimensional Dirac delta function due to complexity of the spectrum, the average is taken over the random measure (2).

Many authors have studied the spectral density (3) in the large N limit [5, 7, 26]. In particular, convenient quaternionic/hermitization methods [15, 25] were developed to complete this task. For $L = R = 1$ and a general normal source S, spectral density in the large–N limit was found in Ref. [27] whereas the $L, R \neq 1$ generalization was recently studied in Ref. [3]. For finite matrix size, a formula for the spectral density was calculated in Ref. [23] for $L = R = 1$ and a normal source term S only. In this work we address the cases $L, R \neq 1$ as well as non–normal S.

A. Generating function

To find the spectral density, we define the averaged ratio of determinants

$$\mathcal{R}_{L,R}(Z, V) = \left\langle \frac{\det(Z - \mathcal{M})}{\det(V - \mathcal{M})} \right\rangle_P \tag{4}$$

with the $2N \times 2N$ block matrices

$$\mathcal{M} = \begin{pmatrix} 0 & M \\ M^\dagger & 0 \end{pmatrix}, \tag{5}$$

$$Z = \begin{pmatrix} L^2 w & z\mathbf{1}_N \\ \bar{z}\mathbf{1}_N & -R^2\bar{w} \end{pmatrix}, V = \begin{pmatrix} L^2 u & v\mathbf{1}_N \\ \bar{v}\mathbf{1}_N & -R^2\bar{u} \end{pmatrix}, \tag{6}$$

where $\mathbf{1}_N$ denotes the $N \times N$ unit matrix. We notice that the matrices Z and V depend on the complex variables z, u, v and w. For $u = w = 0$ we recover the special case

$$\mathcal{R}_{L,R}(z, v) = \left\langle \frac{\det[(z - M)(\bar{z} - M^\dagger)]}{\det[(v - M)(\bar{v} - M^\dagger)]} \right\rangle_P. \tag{7}$$

Although the variables u, w have an interesting interpretation in terms of the eigenvectors [12], we only use their regulatory properties – as long as $u, w \neq 0$, the ratio is finite for all complex v. Importantly, the spectral density

is generated by taking proper derivatives of the averaged ratio, equation

$$\rho(z,\bar{z}) = -\frac{1}{N\pi}\lim_{w\to 0}\frac{\partial}{\partial \bar{z}}\lim_{V\to Z}\frac{\partial}{\partial v}\mathcal{R}_{L,R}(Z,V) \qquad (8)$$

introduced in Ref. [17] for $L = R = 1$.

As a first step we make the chage of variables $Y = LXR$ implying $M = S + Y$ as well as $\mathcal{M} = \mathcal{S} + \mathcal{Y}$. The measure $P(X)dX$ now reads

$$P_{L,R}(Y)dY = C_{L,R}^{-1}\exp\left(-n\mathrm{Tr}R^{-2}Y^\dagger L^{-2}Y\right)dY, \quad (9)$$

where the normalization constant is given as $C_{L,R} = (\pi/n)^{N^2}\det(LR)^2$. We open the ratio of determinants with the help of complex Grassmann variables χ_i and complex ordinary variables ϕ_i,

$$\frac{\det(Z-\mathcal{M})}{\det(V-\mathcal{M})} = c\int d[\phi,\chi]e^{iq^\dagger \mathrm{diag}(V-\mathcal{M},Z-\mathcal{M})q}, \quad (10)$$

with a proper normalization constant c. We introduced the supervector $q = (\phi_1\phi_2\chi_1\chi_2)^T$, and the joint measure $d[\phi,\chi] = \prod_{i=1}^N d(\phi_1)_i d(\phi_2)_i d(\chi_1)_i d(\chi_2)_i$. Averaging with the distribution $P_{L,R}$ only affects the exponential terms proportional to Y which are given by

$$e^{-iq^\dagger \mathrm{diag}(\mathcal{Y},\mathcal{Y})q} = e^{-i\left(\phi_1^\dagger Y\phi_2 + \chi_1^\dagger Y\chi_2 + \phi_2^\dagger Y^\dagger\phi_1 + \chi_2^\dagger Y^\dagger\chi_1\right)} =$$
$$= e^{-i\mathrm{Tr}(E_1Y + E_2Y^\dagger)},$$

where we set $(E_1)_{ij} = (\phi_2)_i(\bar{\phi}_1)_j - (\chi_2)_i(\bar{\chi}_1)_j$ and $(E_2)_{ij} = (\phi_1)_i(\bar{\phi}_2)_j - (\chi_1)_i(\bar{\chi}_2)_j$. The average is easily found to be

$$\int dY P_{L,R}(Y)e^{-i\mathrm{Tr}(E_1Y+E_2Y^\dagger)} = e^{-\frac{1}{n}\mathrm{Tr}E_1L^2E_2R^2}. \quad (11)$$

To proceed further, we carry out a Hubbard–Stratonovich transformation

$$e^{-\frac{1}{n}\mathrm{Tr}E_1L^2E_2R^2} = c_0\int [d\Sigma]e^{-nF-q^\dagger Qq}, \qquad (12)$$

which reduces the fourth order supervector terms to second order. The supermatrix Q appearing in the exponent is given by

$$Q = \begin{pmatrix} \mathcal{L}\mathrm{diag}(\sigma\mathbf{1}_N, -\bar{\sigma}\mathbf{1}_N) & \mathcal{L}\mathrm{diag}(\alpha\mathbf{1}_N, \beta\mathbf{1}_N) \\ \mathcal{L}\mathrm{diag}(\bar{\alpha}\mathbf{1}_N, \bar{\beta}\mathbf{1}_N) & \mathcal{L}\mathrm{diag}(\bar{\rho}\mathbf{1}_N, \rho\mathbf{1}_N)\end{pmatrix}, \quad (13)$$

with $\mathcal{L} = \mathrm{diag}(L^2, R^2)$. It depends on four new complex integration variables, two ordinary ones σ and ρ as well as two anticommuting ones α and β. The corresponding measure

$$[d\Sigma] = d^2\sigma d^2\rho d^2\alpha d^2\beta \qquad (14)$$

is flat. We use the notation $d^2\alpha = d\alpha d\bar{\alpha}$. The normalization constant in Eq. (12) is given by $c_0 = \pi^{-2}$. The function $F = |\sigma|^2 + |\rho|^2 + \bar{\alpha}\beta + \bar{\beta}\alpha$ in the exponent yield the Gaussians needed bring the supervector q to second order.

Thus, we can cast the generating function $\mathcal{R}_{L,R}$ into the form

$$\mathcal{R}_{L,R} = cc_0\int d[\phi,\chi]\int [d\Sigma]e^{-nF+iq^\dagger Aq}, \qquad (15)$$

where we introduced the supermatrix

$$A = \mathrm{diag}(V-\mathcal{S}, Z-\mathcal{S}) + iQ\ . \qquad (16)$$

In the next step we interchange the order of integration $d[\phi,\chi] \leftrightarrow [d\Sigma]$. This, however, has a subtle flaw: the resulting integral in the bosonic σ,ρ directions is no longer convergent, an issue addressed previously [22, 24]. To circumvent this problem, we make the change of variables

$$\rho = \rho_1 + i\rho_2, \qquad \sigma = \sigma_1 + i\sigma_2,$$
$$\rho_1 = i\frac{w-\bar{w}}{2} + f\cos\phi, \qquad \rho_2 = -\frac{w+\bar{w}}{2} + f\sin\phi,$$
$$\sigma_1 = i\frac{u+\bar{u}}{2} - ig_-\sinh\gamma, \qquad \sigma_2 = \frac{u-\bar{u}}{2} + g_-\cosh\gamma,$$

before swapping the order of integration. Here, we introduced real commuting variables f, g, γ and ϕ as well as a small imaginary increment, $g_- = g - i\epsilon$ with $\epsilon > 0$. The range of integration is $f \geq 0, \phi \in (0, 2\pi], g \in \mathbb{R}, \gamma \in \mathbb{R}$. The anticommuting variables α, β remain unchanged. The integral then becomes

$$\int [d\Sigma]e^{-nF+iq^\dagger Aq} = \int [d\Sigma'](-ig_-f)e^{-nF'+iq^\dagger A'q}, \quad (17)$$

with $[d\Sigma'] = dfd\phi dgd\gamma d^2\alpha d^2\beta$ and

$$F' = g_-^2 + f^2 + |w|^2 - |u|^2 + g_-(ue^\gamma - \bar{u}e^{-\gamma}) \qquad (18)$$
$$+ if(we^{i\phi} - \bar{w}e^{-i\phi}) + \bar{\alpha}\beta + \bar{\beta}\alpha.$$

We also introduced the transformed supermatrix

$$A' = \begin{pmatrix} A'_{BB} & A'_{BF} \\ A'_{FB} & A'_{FF}\end{pmatrix}, \qquad (19)$$

with the $2N \times 2N$ blocks

$$A'_{BB} = \begin{pmatrix} -L^2\sigma_-e^{-s} & v\mathbf{1}_N - S \\ \bar{v}\mathbf{1}_N - S^\dagger & -R^2\sigma_-e^s\end{pmatrix}, \quad A'_{BF} = \begin{pmatrix} i\alpha L^2 & 0 \\ 0 & i\beta R^2\end{pmatrix},$$
$$A'_{FF} = \begin{pmatrix} iL^2\rho e^{-i\phi} & z\mathbf{1}_N - S \\ \bar{z}\mathbf{1}_N - S^\dagger & iR^2\rho e^{i\phi}\end{pmatrix}, \qquad A'_{FB} = \begin{pmatrix} i\bar{\alpha}L^2 & 0 \\ 0 & i\bar{\beta}R^2\end{pmatrix}.$$

After this change of variables, we now may safely interchange the order of integration and arrive at

$$\mathcal{R}_{L,R} = -ic_0\int [d\Sigma']g_-fe^{-nF'}\mathrm{sdet}^{-1}A', \qquad (20)$$

where the integral over the supervector yielded the superdeterminant as an extension of Eq. (10)

$$c\int d[\phi,\chi]e^{iq^\dagger A'q} = \mathrm{sdet}^{-1}A'. \qquad (21)$$

The superdeterminant is known to satisfy the formula

$$\mathrm{sdet}^{-x}A' = \frac{\det{}^x A'_{FF}}{\det{}^x A'_{BB}}\left(1 + x\mathrm{Tr}A_0 + \frac{x}{2}\mathrm{Tr}A_0^2 + \frac{x^2}{2}\left(\mathrm{Tr}A_0\right)^2\right)$$

where $A_0 = {A'_{BB}}^{-1}A'_{BF}{A'_{FF}}^{-1}A'_{FB}$ for any integer x. This result enables us to integrate over the Grassmann variables α, β in Eq. (20). The integral

$$I(f,g,\phi,\gamma) = \int d\alpha d\beta e^{-n(\bar{\alpha}\beta+\bar{\beta}\alpha)}\mathrm{sdet}^{-1}A' \qquad (22)$$

can be written in the form

$$I = -G\left(g_1 + (n-g_2)(n-g_3) + g_4\right), \qquad (23)$$

after some algebra and by utilizing the standard normalization of the Berezin integrals to one. The individual terms are

$$G = \frac{\det(-f^2\mathbf{1}_N - \Gamma_z\Omega_z)}{\det(g_-^2\mathbf{1}_N - \Gamma_v\Omega_v)},$$
$$g_2 = \mathrm{Tr}\left[\Omega_z\Gamma_v\mathbf{P}_v\mathbf{Q}_z\right], \quad g_3 = \mathrm{Tr}\left[\Omega_v\Gamma_z\mathbf{Q}_z\mathbf{P}_v\right],$$
$$g_1 = f^2g_-^2\,\mathrm{Tr}\left[\mathbf{P}_v\mathbf{Q}_z\right]\mathrm{Tr}\left[\mathbf{P}'_v\mathbf{Q}'_z\right],$$
$$g_4 = f^2\mathrm{Tr}\left[\Omega_v\mathbf{Q}'_z\Gamma_v\mathbf{P}_v\mathbf{Q}_z\mathbf{P}_v\right] + g_-^2\,\mathrm{Tr}\left[\Omega_z\mathbf{P}'_v\Gamma_z\mathbf{Q}_z\mathbf{P}_v\mathbf{Q}_z\right],$$

where we defined

$$\Omega_x = R^{-2}(\bar{x}\mathbf{1}_N - S^\dagger), \qquad \Gamma_x = L^{-2}(x\mathbf{1}_N - S),$$
$$\mathbf{P}_v = (g_-^2\mathbf{1}_N - \Omega_v\Gamma_v)^{-1}, \quad \mathbf{P}'_v = (g_-^2\mathbf{1}_N - \Gamma_v\Omega_v)^{-1},$$
$$\mathbf{Q}_z = (-f^2\mathbf{1}_N - \Omega_z\Gamma_z)^{-1}, \quad \mathbf{Q}'_z = (-f^2\mathbf{1}_N - \Gamma_z\Omega_z)^{-1}.$$

At this point we make the remarkable observation that the function I is independent of the variables γ and ϕ such that $I(f,g,\phi,\gamma) = I(f,g)$. Hence integrating over the fermionic variables effectively restores a certain invariance.

Assembling everything, the generating function (20) is given by

$$\mathcal{R}_{L,R} = -\frac{4i}{\pi}e^{-n|w|^2+n|u|^2}\int_{-\infty}^{\infty}dg_-\int_0^\infty df J(f,g_-), \qquad (24)$$

with the integrand

$$J(f,g_-) = g_-fe^{-n(g_-^2+f^2)}I(f,g_-)I_0(2nf|w|)K_0(2in|u|g_-), \qquad (25)$$

depending on the modified Bessel functions I_0 and K_0 of the first and second type, respectively. They result from the following integrals over the γ, ϕ variables,

$$N_\gamma = \int_{-\infty}^{\infty}d\gamma e^{-ng_-(ue^\gamma - \bar{u}e^{-\gamma})},$$
$$N_\phi = \int_0^{2\pi}d\phi e^{-inf(we^{i\phi} - \bar{w}e^{-i\phi})}.$$

We set $u = |u|e^{i\theta}, w = |w|e^{i\psi}$ and choose the argument of u to be $\theta = \pi/2$ to make the γ integral convergent. The angle of w is arbitrary since the ϕ integral is periodic. We therefore set $\psi = 0$ and arrive at

$$N_\gamma = \int_{-\infty}^{\infty}d\gamma e^{-2ing_-|u|\cosh\gamma} = 2K_0(2in|u|g_-),$$
$$N_\phi = \int_0^{2\pi}d\phi e^{2nf|w|\sin\phi} = 2\pi I_0(2nf|w|),$$

which after taking care of the constants yields Eq. (24).

III. PARTICULAR CASES

So far, the result (24) for the generating function is exact for any matrix dimension N and is valid for any structural matrices L, R and S. Although the integrand (25) is, in general, rather complicated, the integral can be worked out explicitly for certain subclasses of L, R and S. We are partcularly interested in the three cases

1. normal source S and variance matrices L, R arbitrary,
2. vanishing source $S = 0$ and trivial $L = R = 1$,
3. non–normal source S of rank one and trivial variance matrices $L = R = 1$,

which we compute and discuss in the sequel.

A. Normal S and arbitrary L, R

In this case all structure matrices L, R and S are diagonal,

$$S = \mathrm{diag}(\underbrace{s_1,...,s_1}_{u_1},\underbrace{s_2,...,s_2}_{u_2},\underbrace{...,s_x}_{...u_x}),$$
$$L = \mathrm{diag}(\underbrace{l_1,...,l_1}_{v_1},\underbrace{l_2,...,l_2}_{v_2},\underbrace{...,l_y}_{...v_y}),$$
$$R = \mathrm{diag}(\underbrace{r_1,...,r_1}_{w_1},\underbrace{r_2,...,r_2}_{w_2},\underbrace{...,r_z}_{...w_z}),$$

with three sets of multiplicities u_i, v_i, w_i which should not be confused with the above employed complex variables u, v, w. Here, x, y, z are the numbers of different entries in the structure matrices L, R and S, respectively, therby defining the sizes of the sets. The multiplicities in each set add up to N. Because the integrand (23) only depends on the products $(\Omega_x)_{ii}(\Gamma_y)_{ii}$, we introduce a structured source matrix of the form

$$\alpha_{xy} = \Omega_x\Gamma_y = (LR)^{-2}(\bar{x}\mathbf{1}_N - S^\dagger)(y\mathbf{1}_N - S), \qquad (26)$$

which depends on all three matrices L, R and S. It is accompanied by a merged multiplicity vector $\vec{n}$. We define it by the following construction: we first form the

multiplicity vectors $\vec{u} = (u_1, ..., u_x)$, $\vec{v} = (v_1, ..., v_y)$ and $\vec{w} = (w_1, ..., w_z)$ corresponding to the matrices S, L and R, respectively. The vectors $\vec{u}$ is graphically represented by a column of N points which are ordered in x groups according to the multiplicities u_i. The points within each of these x groups are given the same (arbitrary) color which is only used to distinguish the different groups. We refer to the first and last points in each group as boundary. The vectors $\vec{v}, \vec{w}$ are represented accordingly. The multiplicity vector $\vec{n} = (n_1, ..., n_k)$ is then constructed as a vector which has a boundary whenever *at least* one of the vectors $\vec{u}, \vec{v}$ and $\vec{w}$ has one. We illustrate this by the example in Fig. 1 in which the vector $\vec{u}$ is represented by $N = 11$ points ordered in $x = 3$ groups with multiplicities $u_1 = 5$, $u_2 = 2$ and $u_3 = 4$ with $5 + 2 + 4 = 11$. As seen, the multiplicities for the other two vectors differ. We juxtapose the point sets of all three multiplicity vectors along with the constructed $\vec{n}$. From now on we only

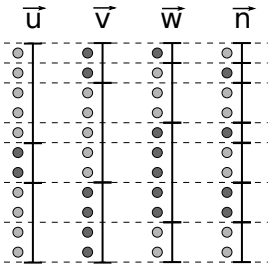

FIG. 1. Construction of the multiplicity vector $\vec{n} = (1, 1, 2, 1, 2, 2, 2)$ from $\vec{u} = (5, 2, 4), \vec{v} = (2, 5, 4), \vec{w} = (1, 3, 5, 2)$. The points depict groups of sizes determined by the corresponding multiplicities. Horizontal lines (both solid and dashed) are drawn along the boundaries of the groups of any of the vectors $\vec{u}, \vec{v}$ and $\vec{w}$, visualizing the construction of the merged vector $\vec{n}$.

use the merged vector $\vec{n}$. We introduce the dimension $d(\vec{n})$ of the vector $\vec{n}$ as the number of differing groups, *e.g.* $d(\vec{n}) = 7$ in the above example. We also introduce the length $|\vec{n}| = \sum_{i=1}^{d(\vec{n})} n_i$. The generating function can then be cast into the form

$$\frac{1}{C}\mathcal{R}_{L,R} = i_{\vec{n}} j_{\vec{n}} - \sum_{i=1}^{d(\vec{n})} \frac{n}{n_i}\left(\alpha^i_{zv} + \alpha^i_{vz} + \frac{N}{n}\right) i_{\vec{n}-\vec{e}_i} j_{\vec{n}+\vec{e}_i} + \\ + \sum_{i,j=1}^{d(\vec{n})} \frac{n^2 \alpha^i_{zv}}{n_i n_j}\Big[\left(\alpha^j_{vz} - \alpha^i_{vz}\right) i_{\vec{n}-\vec{e}_i-\vec{e}_j} j_{\vec{n}+\vec{e}_i+\vec{e}_j}\Big] + \\ + \sum_{i,j=1}^{d(\vec{n})} \frac{n}{n_j}\Big[\alpha^i_{vv} i_{\vec{n}-\vec{e}_j} j_{\vec{n}+\vec{e}_i+\vec{e}_j} + \alpha^i_{zz} i_{\vec{n}-\vec{e}_i-\vec{e}_j} j_{\vec{n}+\vec{e}_j}\Big], \tag{27}$$

where α^i_{xy} is the i–th element of the diagonal matrix (26), $C = \prod_{i=1}^{d(\vec{n})} n_i$, and the $\vec{e}_i$'s are k–dimensional unit vectors in the i–th direction. These vectors $\vec{e}_i$ are used to conveniently add or subtract a single source from the vector $\vec{n}$. The result (27) contains two functions which can be traced back to the Berezin and the ordinary integrals, We refer to them as fermionic and as bosonic building blocks. The former is given by

$$i_{\vec{m}}(z, w) = \frac{e^{-n|w|^2}}{\prod_{i=1}^{d(\vec{m})} m_i!} \int_0^\infty d\rho e^{-\rho} I_0(2\sqrt{n\rho}|w|) \prod_{i=1}^{d(\vec{m})} \left(\rho + n\alpha^i_{zz}\right)^{m_i}, \tag{28}$$

where we set $i_{\vec{m}} = 0$ if some element of the multiplicity vector $\vec{m}$ is negative. The bosonic counterpart reads

$$j_{\vec{m}}(v, u) = \frac{2in}{\pi} \prod_{i=1}^{d(\vec{m})} \frac{(m_i - 1)!}{(-n)^{m_i}} e^{n|u|^2} \times \\ \times \int_{-\infty}^{\infty} dg g_- e^{-n g_-^2} K_0(2in|u|g_-) \prod_{i=1}^{d(\vec{m})} (g_-^2 - \alpha^i_{vv})^{-m_i}. \tag{29}$$

We notice that the bosonic building block may be expressed as the contour integral

$$j_{\vec{m}}(v, u) = \frac{\prod_{i=1}^{d(\vec{m})}(m_i - 1)!}{2\pi i} \oint_{\Gamma_s} dp \sum_{k=0}^{\infty} \frac{U_{k+1,1}(n|u|^2) p^k}{\prod_{i=1}^{d(\vec{m})} \left(p + n\alpha^i_{vv}\right)^{m_i}}, \tag{30}$$

where the contour Γ_s encircles all sources $-n\alpha^i_{vv}$ counterclockwise. Here, $U_{a,b}(z) = U(a, b, z)$ is the Tricomi confluent hypergeometric function. Details of the calculation are provided in the App. A.

Before proceeding we cross–check the generating function (27) with similar calculations carry out for the chiral Gaussian Unitary Ensemble. Choosing the trivial covariance $L = R = 1$ and a vanishing source $S = 0$ at the origin $z = v = 0$ the generating function reduces to

$$\mathcal{R}_{\text{chGUE}} = \left\langle \frac{\det(|w|^2 + XX^\dagger)}{\det(|u|^2 + XX^\dagger)} \right\rangle_P . \tag{31}$$

We also set $n = N$ and arrive at

$$\mathcal{R}_{\text{chGUE}} = N\left(i_N(w) j_N(u) - i_{N-1}(w) j_{N+1}(u)\right),$$

where the index N is a short–hand notation for the one–dimensional multiplicity vector $\vec{n} = (N)$. We find from the formulas (28) and (30) for the fermionic and bosonic building blocks

$$i_m(w) = L_m(-N|w|^2), \quad j_m(u) = (x - 1)! U_{m,1}(N|u|^2),$$

which reproduces the results of Ref. [19]. However, in the present study we are interested in the complementary limit, *i.e.*, we set $u, w \to 0$ and look at $z, v \neq 0$.

We now wish to calculate the spectral density. We recall the formula (8) where the parameters u and w serve as regulators. It is desirable to set them to zero before computing the derivatives. Even though this does not pose a problem for the fermionic block (28), it turns out

to produce infinities in the bosonic block (30). To control these emerging singularities, we use the identity

$$k!U_{k+1,1}(n|u|^2) = e^{n|u|^2}\Gamma(0,n|u|^2)L_k(-n|u|^2) + \\ + \tilde{L}_k(-n|u|^2),$$

for the confluent hypergeometric function. Here, L_k are the Laguerre polynomials whereas $\tilde{L}_k$ are defined by the same recurrence relations but with different initial conditions $\tilde{L}_0(x) = 0, \tilde{L}_1(x) = -1$. The singular behavior for U as $u \to 0$ is due to the incomplete Gamma function $\Gamma(0, n|u|^2)$ in the first term. We therefore split the bosonic block into a singular and a regular parts,

$$j_{\vec{m}}(v,u) = j^{(\mathrm{sing})}_{\vec{m}}(v,u) + j^{(\mathrm{reg})}_{\vec{m}}(v,u). \tag{32}$$

To control the singularity, we set the singular part $j^{(\mathrm{sing})}$ to zero and take the limit $u \to 0$ in the regular part $j^{(\mathrm{reg})}$. We formalize this procedure by introducing the regularized generating function

$$\tilde{\mathcal{R}}_{L,R} = \mathcal{R}_{L,R}\left[i_{\vec{m}}(z,w) \to \tilde{i}_{\vec{m}}(z), j_{\vec{m}}(v,u) \to \tilde{j}_{\vec{m}}(v)\right], \tag{33}$$

with new building blocks $\tilde{i}_{\vec{m}}(z) = i_{\vec{m}}(z, w = 0)$ and $\tilde{j}_{\vec{m}}(v) = j^{(reg)}_{\vec{m}}(v, u = 0)$ already in the $w, u \to 0$ limit. We stress that this procedure is not an approximation — although we have $\tilde{\mathcal{R}}_{L,R} \neq \mathcal{R}_{L,R}$, the spectral densities obtained by Eq. (8) agree exactly $\tilde{\rho} = \rho$. We checked this numerically. This property is intuitively justified since we subtract the otherwise infinite part proportional to $j^{(\mathrm{sing})}$. The regularized building blocks are given by

$$\tilde{i}_{\vec{m}} = \frac{1}{\prod_{i=1}^{d(\vec{m})} m_i!}\int_0^\infty d\rho e^{-\rho}\prod_{i=1}^{d(\vec{m})}\left(\rho + n\alpha^i_{zz}\right)^{m_i}, \\ \tilde{j}_{\vec{m}} = -\frac{\prod_{i=1}^{d(\vec{m})}(m_i-1)!}{2\pi i}\oint_{\Gamma_s} dp \frac{e^p(\gamma + \Gamma(0,p) + \ln p)}{\prod_{i=1}^{d(\vec{m})}(p + n\alpha^i_{vv})^{m_i}}, \tag{34}$$

where we used the identity

$$\sum_{m=0}^{\infty}\frac{1}{m!}\tilde{L}_m(0)p^m = -e^p(\gamma + \Gamma(0,p) + \ln p) \tag{35}$$

for the modified Laguerre polynomials with γ denoting the Euler constant. This identity follows from the fact that $\tilde{L}_m(0) = -\sum_{k=1}^m \frac{1}{k}$ are the (negative) harmonic numbers.

The final formula for the spectral density in the case of a normal source S and nontrivial L, R then reads

$$\tilde{\rho} = -\frac{1}{N\pi}\frac{\partial}{\partial \bar{z}}\lim_{V\to Z}\frac{\partial}{\partial v}\tilde{\mathcal{R}}_{L,R}(z,v), \tag{36}$$

together with the definitions (27), (33) and (34). We demonstrate the utility of our analytical result in Fig. 2 by comparing it with numerical simulations. Adding

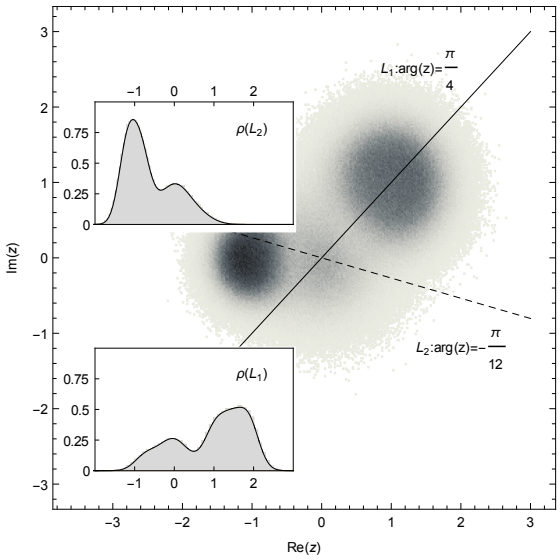

FIG. 2. Spectral density according to Eq. (36) as insets along two lines L_1 and L_2 in the complex plane, together with numerical simulations. The structural matrices are $S = \mathrm{diag}(-1, 0, 1+i)$, $L = \mathrm{diag}(3/4, 1)$ and $R = \mathrm{diag}(1, 5/4, 1)$ with multiplicity vectors of $\vec{u} = (2,1,3)$, $\vec{v} = (2,4)$ and $\vec{w} = (2,1,3)$.

(structured) noise LXR produces an overall eigenvalues spreading with anisotropic features reflecting the L, R covariance matrices. The density is concentrated around the initial eigenvalues of S and varies smoothly as we change the noise level n, *i.e.* the inverse variance of the ensemble (2).

B. Vanishing source $S = 0$ and $L = R = 1$

We now consider the case $S = 0$ and $L = R = 1$ in which a simple spectral density formula is known from the work of Ginibre [20]. The multiplicity vector is one–dimensional $\vec{n} = (N)$ and the source matrix has the simple form $\alpha_{xy} = \bar{x}y\mathbf{1}_N$. The regularized generating function (33) reads

$$\tilde{\mathcal{R}}_G = N(\tilde{i}_N\tilde{j}_N - \tilde{i}_{N-1}\tilde{j}_{N+1}) - n\tilde{i}_{N-1}\tilde{j}_{N+1}(\bar{v}z + \bar{z}v) + \\ + n\left(\tilde{i}_{N-1}\tilde{j}_{N+2}|v|^2 + \tilde{i}_{N-2}\tilde{j}_{N+1}|z|^2\right), \tag{37}$$

where we write $\tilde{i}_N = \tilde{i}_{\vec{n}}$, $\tilde{j}_N = \tilde{j}_{\vec{n}}$. The building blocks are

$$\tilde{i}_\alpha = \frac{1}{\alpha!}\int_0^\infty d\rho e^{-\rho}(\rho + n|z|^2)^\alpha, \tag{38}$$

$$\tilde{j}_\beta = -\frac{(\beta-1)!}{2\pi i}\oint_\Gamma dp\frac{e^p \ln p}{(p + n|v|^2)^\beta}. \tag{39}$$

The bosonic block, when compared to Eq. (34), lacks the term $\gamma + \Gamma(0,p)$ since this contribution vanishes in the generating function (33), as can be seen by a symbolic

calculation. This observation holds more generally, not only in this simplest case. Directly from the definitions, we derive the iterative formulas

$$\tilde{i}_\alpha = \tilde{i}_{\alpha-1} + (n|z|^2)^\alpha(\alpha!)^{-1},$$
$$\tilde{j}_\beta = \tilde{j}_{\beta+1} - (\beta-1)!(n|v|^2)^{-\beta}e^{-n|v|^2}\tilde{i}_{\beta-1}(v)$$

and use them to re–express the generating function

$$\tilde{\mathcal{R}}_G = n\tilde{i}_{N-1}\tilde{j}_{N+1}|v-z|^2 + \\ + \frac{e^{-n|v|^2}}{|v|^{2N}}\left(\tilde{i}_{N-1}(z)|v|^{2N} - \tilde{i}_{N-1}(v)|z|^{2N}\right), \quad (40)$$

where we have written out explicitly the argument of $\tilde{i}$ to avoid confusion. At this point we observe that the generating function vanishes for $z = v$, $\tilde{\mathcal{R}}_G = 0$. It is thus evident that the derivative formula (36) only produces contributions due to the second term. Lastly, by using $\partial_{\bar{z}}\tilde{i}_\alpha = nz\tilde{i}_{\alpha-1}$ and $\partial_v\tilde{j}_\beta = -n\bar{v}\tilde{j}_{\beta+1}$, we recover the well–known formula

$$\rho_G = \frac{n}{N\pi}e^{-n|z|^2}\sum_{k=0}^{N-1}\frac{(n|z|^2)^k}{k!}, \quad (41)$$

for the spectral density, which often appears for $n = N$.

C. Non–normal rank–1 S and $L = R = 1$

A major reason to study models of the type (1) is the issue of spectral stability. — How far do the eigenvalues of $S + Y$ spread around the eigenvalues of S for a small perturbation Y. This is especially interesting for finite rank sources S where extremal (or outlier) eigenvalues emerge from the eigenvalue sea of the matrix Y. This phenomenon was studied in a Hermitian [6, 13, 33] as well as a non–Hermitian [32, 36, 37] setting. Here, we examine how the normal or non–normal character of the source influences the eigenvalue distribution. We consider a rank-one source of the form

$$S = \alpha\,|n\rangle\langle m|\,, \quad (42)$$

for complex parameter α and bras (kets) $\langle m|$ ($|n\rangle$) denoting the canonical matrix basis – the source matrx S has one non–zero element α placed on the off–diagonal. For the sake of simplicity we choose the trivial variance structure $L = R = 1$. After a fair amount of algebra we find the result

$$\mathcal{R}_{NN} = R_0 + |\alpha|^2R_1 + |\alpha|^4R_2 + |\alpha|^6R_3 + |\alpha|^8R_4 \quad (43)$$

for the generating function. The formulas for the R_i's are lengthy and thus were explicitly given only in the App. B. Although the terms in Eq. (43) turn out to lack structure, they are still assembled from the bosonic and fermionic building blocks similar to Eq. (28),

$$i_{k,l}(z,w) = \frac{(-1)^k}{n^{k+2l+1}}e^{-n|w|^2}\int_0^\infty d\rho e^{-\rho}I_0(2\sqrt{n\rho}|w|)\times \\ \times(\rho+n|z|^2)^k(\rho+nk_z^+)^l(\rho+nk_z^-)^l, \quad (44)$$

and Eq. (29),

$$j_{q,r}(v,u) = \frac{2}{i\pi}e^{n|u|^2}\int_{-\infty}^\infty dgg_-e^{-ng_-^2}K_0(2in|u|g_-)\times \\ \times(g_-^2-|v|^2)^{-q}(g_-^2-k_v^+)^{-r}(g_-^2-k_v^-)^{-r}, \quad (45)$$

where $k_x^\pm = \frac{1}{2}\left(|\alpha|^2+2|x|^2 \pm |\alpha|\sqrt{4|x|^2+|\alpha|^2}\right)$. By investigating the terms in each of the R_i's, we find the conditions $l = -1, 0, 1$, $k \geq 0$ and $q + r \geq 1$, $r = 1, 2, 3$. for the indices of $i_{k,l}$ and $j_{q,r}$, respectively. We employ the same regularization steps as in Sec. III A, obtain the generating function $\tilde{\mathcal{R}}_{NN}$ and construct the regularized fermionic block

$$\tilde{i}_{k,0} = \frac{(-1)^k k!}{n^{k+1}}(\tilde{i}_G)_k, \quad (46)$$
$$\tilde{i}_{k,1} = \tilde{i}_{k+2,0} - |\alpha|^2(\tilde{i}_{k+1,0} + |z|^2\tilde{i}_{k,0}), \quad (47)$$
$$\tilde{i}_{k,-1} = \frac{(-1)^k k!}{(k_z^+-k_z^-)n^k}\times \\ \times\sum_{l=0}^k\frac{(n|z|^2)^l}{l!}\Big[U_{1,1+l-k}(nk_z^-) - U_{1,1+l-k}(nk_z^+)\Big], \quad (48)$$

where $\tilde{i}_G$ is the Ginibre block of Eq. (38) and $k \geq 0$. We relegate the derivation of Eq. (48) to the App. B. The bosonic block reads

$$\tilde{j}_{q,r} = -\frac{(-n)^{q+2r-1}}{2\pi i}\oint_\Gamma\frac{dpe^p\ln p}{(p+n|v|^2)^q(p+nk_v^-)^r(p+nk_v^+)^r}, \quad (49)$$

where $q \geq 0, r \geq 1$ and the contour Γ encircles both $-n|v|^2$ and $-nk_v^\pm$. Lastly we obtain the formulas for $q = -1, -2$,

$$\tilde{j}_{-1,2} = \frac{1}{2}\left(\tilde{j}_{0,2_-} + \tilde{j}_{0,2_+} + |\alpha|^2\tilde{j}_{0,2}\right), \quad (50)$$
$$\tilde{j}_{-1,3} = \frac{1}{2}\left(\tilde{j}_{0,3_-} + \tilde{j}_{0,3_+} + |\alpha|^2\tilde{j}_{0,3}\right), \quad (51)$$
$$\tilde{j}_{-2,3} = \frac{1}{4}\left(\tilde{j}_{0,3_{--}} + 2\tilde{j}_{0,3_{+-}} + \tilde{j}_{0,3_{++}} + |\alpha|^4\tilde{j}_{0,3} + \right. \\ \left. +2|\alpha|^2(\tilde{j}_{0,3_+} + \tilde{j}_{0,3_-})\right), \quad (52)$$

where the subscripts $\pm$ indicate that the underlying multiplicity vector $\vec{x} = (q, r-1, r)$ is applied with decrement to the source at $nk_v^\pm$.

Finally, we obtain the spectral density (3) analytically and plot it in Fig. 3. To facilitate a comparison, we juxtapose it with the analogous results for the case of a rank–one normal source S and for the Ginibre case (41). A non–normal source S (third row in Fig. 3) does not produce, on average, outlier eigenvalues in the spectrum, in contrast normal source S (second row in Fig. 3) where we find an island around $\alpha = 10$. Instead, in the non–normal case we observe something like a blow–up of the spectral bulk. The first row in Fig. 3 is devoted to the case of a vanishing source, $S = 0$. Near $z = 0$ both,

normal and vanishing source, produce similarly shaped spectral densities — the only difference between these cases is the presence or absence of the finite–rank island.

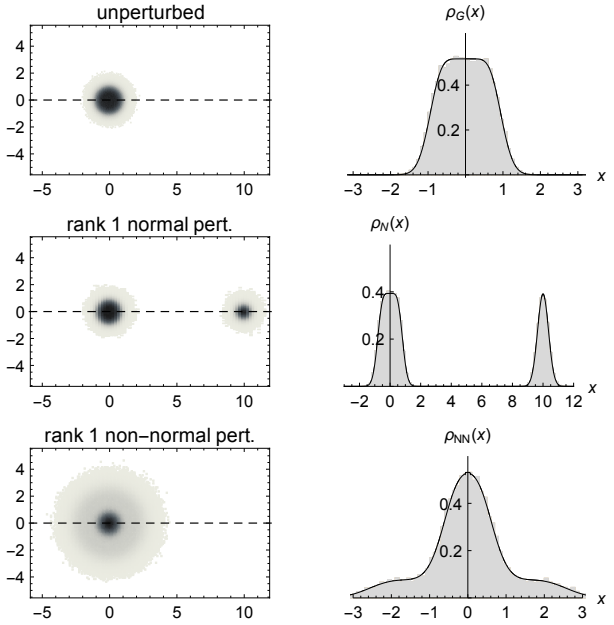

FIG. 3. Left hand side: complex plane of eigenvalues, from top to bottom for: unperturbed $S = 0$ (Ginibre), normal perturbation $S = 10\,|1\rangle\langle 1|$ and non–normal perturbation $S = 10\,|2\rangle\langle 1|$. Right hand side: numerical simulations and analytical results for the spectral densities ρ_G, ρ_N and ρ_{NN} along the real axis line (dashed lines on the left hand side). Numerical simulations are for matrices of size $N = 4$, $\alpha = 10$, we set $n = N$.

D. Spectrum of M^{-1}

As a last application we discuss how to infer somewhat gratuitously the spectrum of $(S+X)^{-1}$ from the results for the spectrum of $S + X$. For simplicity we deal with a normal source S only and set $L = R = 1$. To this end we define a generating function $\mathcal{R}_{-1}$ for the inverse as

$$\mathcal{R}_{-1}(Z,V) = \left\langle \frac{\det(Z-\mathcal{M}_{-1})}{\det(V-\mathcal{M}_{-1})} \right\rangle = \\ = \frac{\det Z}{\det V}\mathcal{R}_{1,1}\left(Z',V'\right), \qquad (53)$$

and relate it to the generating function (4) previously considered. The matrices $\mathcal{M}_{-1} = \begin{pmatrix} 0 & M^{-1} \\ M^{\dagger,-1} & 0 \end{pmatrix}$ and Z', V' are rearranged versions of the inverse matrices Z^{-1}, V^{-1} of Eq. (6),

$$X' = \begin{pmatrix} (X^{-1})_{22} & (X^{-1})_{21} \\ (X^{-1})_{12} & (X^{-1})_{11} \end{pmatrix}, \quad X = Z, V. \qquad (54)$$

We thus conclude that the whole calculation discussed in Sec. III A can be repeated with only making the replacements $w \to -wG_{zw}, z \to \bar{z}G_{zw}, u \to -uG_{vu}$ and $v \to \bar{v}G_{vu}$ with $G_{xy} = (|x|^2 + |y|^2)^{-1}$. We again conduct the regularization procedure and eventually find that only the source matrix of Eq. (26) is modified according to

$$\alpha_{xy} \to (\alpha^{-1})_{xy} = \alpha_{x^{-1}y^{-1}} = (\bar{x}^{-1}\mathbf{1}_N - S^\dagger)(y^{-1}\mathbf{1}_N - S),$$

The regularized ratio for the problem of finding the spectrum of $(S+X)^{-1}$ reads

$$\tilde{R}_{-1} = \left(\frac{|z|^2}{|v|^2}\right)^{|\vec{n}|} \tilde{\mathcal{R}}_{1,1}\left[\alpha_{xy} \to (\alpha^{-1})_{xy}\right], \qquad (55)$$

where the generating function $\tilde{\mathcal{R}}_{1,1}$ is that of Eq. (33) and the constituent fermionic and bosonic blocks (34) are affected accordingly. In particular, we calculate the spectral density for an inverse matrix X^{-1} as

$$\rho_{G,-1} = \frac{n e^{-\frac{n}{|z|^2}}}{N\pi|z|^4} \sum_{k=0}^{N-1} \frac{1}{(k)!}\left(\frac{n}{|z|^2}\right)^k, \qquad (56)$$

obtained from Eq. (40). This formula was also found in a recent work on the product of matrices [2]. In Fig. 4, the

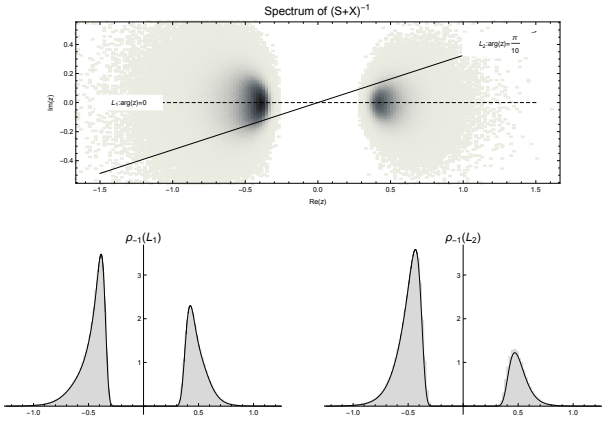

FIG. 4. A numerical simulation along with analytic spectral density plots of matrix $(S+X)^{-1}$ along two straight lines L_1 and L_2 for an external source setup as $S = (-2, 2)$ with multiplicities $\vec{n} = (4, 2)$.

spectral density of $(S+X)^{-1}$ is depicted as calculated from the generating function (55) for non–zero external source S.

IV. CONCLUSIONS

We have calculated exact spectral densities for a class of complex random matrix models of the form $M = S + LXR$ consisting of a noise part X and structure parts

S, L, R. We found two–fold integral formulas for arbitrary structural matrices. In greater detail, we investigated the case of a normal source matrix S and arbitrary diagonal matrices L, R which are of particular interest. The resulting formulas are of a remarkably succinct form. We confirmed our analytical results by numerical simulations.

We showed how the presence or absence of the normality condition for S leads to a qualitatively different behaviour of the eigenvalue densities. Our study was focused mainly on the finite rank source matrices where analytical solutions proved tractable. For a non–normal source, the most interesting feature is the lack of outliers, *i.e.*, extreme values in the averaged spectral density. However, when imposing the normality condition on the source matrix S, the outliers are clearly present in the spectral density.

Lastly, we looked at the problem of finding spectra of an inverse matrix M^{-1} which, by using the approach in this paper, proved to be trivially connected to the spectrum of M.

Among the open problems in the context of our study, the question remains on whether the normal vs. non–normal dichotomy has any counterpart relevant for applications. Secondly, the information on eigenvectors is encoded in the objects of study but was, due to the approach taken, completely omitted in our present work. Thirdly, issues related to universality seem feasible within our approach and are certainly worth future investigation.

V. ACKNOWLEDGEMENTS

We gratefully acknowledge fruitful conversations with M. Kieburg, M. Nowak and T. Wirtz. JG appreciates the hospitality of Duisburg–Essen University where part of this work was done in the framework of the Erasmus+ exchange programme. JG also acknowledges the support of the Grant DEC-2011/02/A/ST1/00119 of the Polish National Centre of Science.

Appendix A: Derivation of (30)

We start from equation (29):

$$j_{\vec{m}}(v,u) = \frac{2in}{\pi}\prod_{i=1}^{k}\frac{(m_i-1)!}{(-n)^{m_i}}e^{n|u|^2}J_{\vec{m}}(v,u), \tag{A1}$$

$$J_{\vec{m}}(v,u) = \int_{-\infty}^{\infty} dg g_- e^{-ng_-^2} K_0(2in|u|g_-)\prod_{i=1}^{k}(g_-^2-\alpha_{vv}^i)^{-m_i}, \tag{A2}$$

where we set $d(\vec{m}) = k$ for brevity. By Lagrange interpolation formula we find:

$$\prod_{i=1}^{k}\left(g_-^2-\alpha_{vv}^i\right)^{-m_i} = \lim_{\gamma_1\ldots\gamma_k\to 1}\sum_{l=1}^{k}\mathcal{D}_l(g_-^2-\gamma_l\alpha_{vv}^l)^{-1},$$

with the operator $\mathcal{D}_l$ defined as

$$\mathcal{D}_l = \prod_{i=1}^{k}\frac{(\alpha_{vv}^i)^{1-m_i}}{(m_i-1)!}\frac{d^{m_i-1}}{d\gamma_i^{m_i-1}}\prod_{j=1(\neq l)}^{k}(\gamma_l\alpha_{vv}^l-\gamma_j\alpha_{vv}^j)^{-1},$$

So that the whole integral $J_{\vec{m}}$ is expressed as

$$J_{\vec{m}} = \lim_{\gamma_1\ldots\gamma_k\to 1}\sum_{l=1}^{k}\mathcal{D}_l C_l. \tag{A3}$$

From now on we focus on the integral C_l:

$$C_l = \int_{-\infty}^{\infty} dg\frac{g_- e^{-ng_-^2}}{g_-^2-\alpha_{vv}^l\gamma_l}K_0(2ni|u|g_-), \tag{A4}$$

We re-introduce the representation $K_0(2ni|u|g_-) = \int_0^\infty ds\exp\left(-2ni|u|g_-\cosh s\right)$ and compute:

$$C_l = \frac{1}{2\sqrt{\gamma_l\alpha_{vv}^l}}\int_0^\infty ds\left(I_+(s)-I_-(s)\right), \tag{A5}$$

$$I_\pm(s) = \int_{-\infty}^{\infty} dg\frac{f(g_-,s)}{g-(\pm\sqrt{\gamma_l\alpha_{vv}^l}+i\epsilon)}, \tag{A6}$$

with $f(x,s) = xe^{-nx^2-2ni|u|x\cosh s}$. The integrals $I_\pm$ are calculable by Sokhotski–Plemelj formula:

$$I_\pm(s) = i\pi f(\pm\sqrt{\gamma_l\alpha_{vv}^l},s) + \mathrm{PV}\int_{-\infty}^{\infty}\frac{dxf(x,s)}{x-(\pm\sqrt{\gamma_l\alpha_{vv}^l})}. \tag{A7}$$

The second part is the Hilbert transform [28]:

$$\frac{1}{\pi}\mathrm{PV}\int_{-\infty}^{\infty}dy\frac{ye^{-ay^2-by}}{y-x} = \frac{1}{\sqrt{a\pi}}e^{b^2/4a}+ \\ + ixe^{-x^2a-xb}\mathrm{erf}\left(\frac{i}{2\sqrt{a}}(b+2ax)\right). \tag{A8}$$

Lastly, we need the identity:

$$\int_x^\infty dte^{-a^2t^2-b^2/t^2} = \\ = \frac{\sqrt{\pi}}{4a}\left(e^{2ab}\mathrm{erfc}(ax+b/x)+e^{-2ab}\mathrm{erfc}(ax-b/x)\right), \tag{A9}$$

valid for $x > 0$. Combining the formulas of (A7)-(A9) result in

$$C_l = 2i\sqrt{\pi n}|u|e^{-n\alpha_{vv}^l\gamma_l}\times \\ \times\int_0^\infty ds\int_1^\infty dt\cosh s\; e^{\frac{n\alpha_{vv}^l\gamma_l}{t^2}-n|u|^2t^2\cosh^2 s}, \tag{A10}$$

In the next step we integrate over s and change variables $t^2 = \tau + 1$:

$$C_l = \frac{i\pi}{2}\int_0^\infty d\tau \frac{1}{\tau+1} e^{-n|u|^2(\tau+1)-n\gamma_l\alpha^l_{vv}\frac{\tau}{\tau+1}}. \quad \text{(A11)}$$

We introduce a succinct contour integral representation:

$$\lim_{\gamma_1\dots\gamma_k\to 1}\sum_{l=1}^k \mathcal{D}_l e^{-n\gamma_l\alpha^l_{vv}\frac{\tau}{\tau+1}} = \frac{1}{2\pi i}\oint_{\Gamma'_s} dq \frac{e^{-nq\frac{\tau}{\tau+1}}}{\prod_{i=1}^k (q-\alpha^i_{vv})^{m_i}},$$

where the contour Γ'_s encircles all α^i_{vv}'s counterclockwise. This formula is a part of (**??**) which, after changing $p = -nq$ is equal to:

$$J_{\vec{m}} = \frac{i\pi}{2}(-n)^{|\vec{m}|-1}e^{-n|u|^2}\times \\ \times \frac{1}{2\pi i}\int_0^\infty d\tau \oint_{\Gamma_s} dp \frac{1}{\tau+1}\frac{e^{-n|u|^2\tau+\frac{p\tau}{\tau+1}}}{\prod_{i=1}^k (p+n\alpha^i_{vv})^{m_i}}, \quad \text{(A12)}$$

with appropriately modified contour Γ_s. Lastly, we use an integral representation of the Tricomi confluent hypergeometric function:

$$\int_0^\infty d\tau \frac{1}{\tau+1} e^{-n|u|^2\tau+\frac{p\tau}{\tau+1}} = \sum_{k=0}^\infty U_{k+1,1}(n|u|^2)p^k,$$

and combine it with (A1) and (A12):

$$j_{\vec{m}} = \frac{\prod_{i=1}^{|\vec{m}|}(m_i-1)!}{2\pi i}\oint_{\Gamma_s} dp \sum_{k=0}^\infty \frac{U_{k+1,1}(n|u|^2)p^k}{\prod_{i=1}^{|\vec{m}|}(p+n\alpha^i_{vv})^{m_i}},$$

which is exactly the formula (30).

Appendix B: Details of non-normal S case

The ratio for non-normal case is given by (43) with R_i terms:

$$\begin{aligned}
R_0 &= 2(Vi_{N-3,1}j_{N,1} + Zi_{N,-1}j_{N-3,2}) + 6(Vi_{N-1,0}j_{N-4,3} + Zi_{N-4,1}j_{N-1,1}) - 4Vj_{N-2,2}\delta_1^+ - 4Zi_{N-2,0}\sigma_1^+ + \\
&+ N^2\left[j_{N-1,1}\Delta^Z_{N-3,1} + Vi_{N-3,1}j_{N,1}\right] + nd_1\left[(N-2)j_{N-1,1}i_{N-3,1} + 2i_{N-1,0}j_{N-3,2}\right] - n^2 i_{N-2,1}j_{N-2,1} + \\
&+ N\left[2Vj_{N-2,2}\delta_1^+ - 2Zj_{N-1,1}\delta_2 + 2j_{N-3,2}\Delta^Z_{N-1,0} - 2j_{N-1,1}\Delta^Z_{N-3,1} - Zi_{N-4,1}j_{N-1,1} - 3Vi_{N-3,1}j_{N,1}\right], \\
R_1 &= -N\left[\delta_1^-\Sigma^V_{N-2,2} + \Delta^Z_{N-2,0}\sigma_1^-\right] + n\left[2\Delta^Z_{N-1,0}\Sigma^V_{N-3,2} + d_2 i_{N-2,0}j_{N-2,2}\right] + i_{N-1,0}(2Vj_{N-1,2} + 3j_{N-4,3}) + \\
&+ d_1\left[2Nj_{N-2,2}\Delta^Z_{N-2,0} + i_{N-2,0}(4Vj_{N-3,3} - Nj_{N-2,2}) + i_{N-2,0}j_{N-4,3} - i_{N,-1}j_{N-2,2} + V(N-2)i_{N-2,0}j_{N-1,2} + \right. \\
&\left. - Z(N+2)i_{N-3,0}j_{N-2,2}\right] + 2Vj_{N-4,3}\delta_3 - Zi_{N,-1}\sigma_2 - 2i_{N-3,1}\Sigma^V_{N-2,2} - 2j_{N-1,1}\Delta^Z_{N-2,0} - 2Zj_{N-3,2}\Delta^Z_{N-1,-1} + \\
&+ 2Vi_{N-1,0}\Sigma^V_{N-3,3} + j_{N-3,2}(2Zi_{N-3,0} + i_{N,-1}) - (Vi_{N-2,0} - Zi_{N,-1})j_{N-4,3}, \\
R_2 &= d_1\left[\Delta^Z_{N-1,-1}j_{N-2,2} + (i_{N-2,0} - 2i_{N,-1})\Sigma^V_{N-3,3}\right] + 2(N-2)\Delta^Z_{N-2,0}\Sigma^V_{N-2,2} + Vi_{N-2,0}\Sigma^{-V}_{N-3,3} + \\
&- 2(Z+V)j_{N-4,3}\Delta^Z_{N-1,-1} - \Sigma^-_{N-3,2}(Z)\Delta^Z_{N-1,-1} - i_{N-1,0}\Sigma^V_{N-3,3} + j_{N-2,2}i_{N-2,0} - \delta_3\sigma_2, \\
R_3 &= -\delta_3\Sigma^V_{N-3,3} + \Delta^Z_{N-1,-1}\left[\sigma_2 + 2d_1\Sigma^V_{N-3,3}\right], \\
R_4 &= \Delta^Z_{N-1,-1}\Sigma^V_{N-3,3},
\end{aligned}$$

where $V = |v|^2, Z = |z|^2, d_1 = \bar{z}v + z\bar{v}, d_2 = (\bar{z}v)^2 + (z\bar{v})^2$ and the notation reads

$$\begin{aligned}
&\delta_1^\pm = i_{N-1,0} \pm i_{N-3,1}, \quad \delta_2 = i_{N-4,1} - i_{N-2,0}, \\
&\delta_3 = i_{N,-1} - i_{N-2,0}, \\
&\sigma_1^\pm = j_{N-3,2} \pm j_{N-1,1}, \quad \sigma_2 = j_{N-2,2} - j_{N-4,3}, \\
&\Delta^z_{x,y} = i_{x,y} + zi_{x-1,y}, \quad \Sigma^z_{x,y} = j_{x,y} + zj_{x+1,y}.
\end{aligned}$$

Now we turn to the calculation of regularized bosonic block $\tilde{i}_{k,-1}$ of (48). We start from the definition (44):

$$i_{k,-1} = \frac{(-1)^k}{n^{k-1}}e^{-n|w|^2}\times \\ \times \int_0^\infty d\rho e^{-\rho} I_0(2\sqrt{n\rho}|w|)\frac{(\rho+n|z|^2)^k}{(\rho+nk_z^+)(\rho+nk_z^-)}.$$

Firstly, we express the denominator as an integral:

$$\frac{1}{(\rho+nk_z^+)(\rho+nk_z^-)}=\frac{1}{2n\delta k}\int_0^\infty dp e^{-p\rho-pnk_0}\sinh(pn\delta k),$$

with $k_z^\pm=k_0\pm\delta k$. We consider the integral:

$$\mathcal{I}(p)=\int_0^\infty d\rho e^{-(1+p)\rho}(\rho+n|z|^2)^k I_0(2\sqrt{n\rho}|w|)=$$
$$=e^{\frac{n|w|^2}{p+1}}\frac{(n|z|^2)^k k!}{p+1}\sum_{l=0}^k\frac{(n|z|^2(p+1))^{-l}}{(k-l)!}L_l\left(-\frac{n|w|^2}{p+1}\right),$$

and obtain the formula for $i_{k,-1}$:

$$i_{k,-1}=\frac{(-1)^k}{2n^k\delta k}e^{-n|w|^2}\int_0^\infty dp e^{-pnk_0}\sinh(pn\delta k)\mathcal{I}(p).$$

It gets simplified in the regularization $w\to 0$ limit:

$$\tilde{i}_{k,-1}=\frac{(-1)^k k!}{2n^k\delta k}\times$$
$$\times\sum_{l=0}^k\frac{(n|z|^2)^l}{l!}\Big[U_{1,1+l-k}(nk_z^-)-U_{1,1+l-k}(nk_z^+)\Big],$$

thus reproducing the equation (48).

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