Exact spectral densities of complex noise-plus-structure random matrices

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We use supersymmetry to calculate exact spectral densities for a class of complex random matrix models having the form $M = S + LXR$, where X is a random noise part X, and S,L,R are fixed structure parts. This is a certain version of the "external field" random matrix models. We find twofold integral formulas for arbitrary structural matrices. We investigate some special cases in detail and carry out numerical simulations. The presence or absence of a normality condition on S leads to a qualitatively different behavior of the eigenvalue densities.

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I. NOISE-PLUS-STRUCTURE RANDOM MATRICES

In the past 50 years, random matrix theory (RMT) has been established as an impressively versatile approach [1] to studying complex systems. In particular, applications including large data structures [2], machine learning algorithms [3], and telecommunications [4] have arisen recently. It is a common problem in these and many other areas to infer a signal or information from noisy data. In this work, we study a type of RMT noise-plus-structure model suitable for this type of inference tasks. More specifically, let M be a matrix of the form

$$M = S + LXR, \tag{1}$$

where S is a fixed matrix and $L,R > 0$ are diagonal positive-definite covariance matrices. The matrix X is the source of noise drawn typically from a multidimensional Gaussian ensemble. Equation (1) thus comprises the simplest model combining both randomness (X) and structure (S,L,R). The matrix S is called a source and is interpreted as the signal-information matrix of the system under study. We add a structured noise LXR as every real-world datum is contaminated, and only the resulting matrix M is attainable by experiment. The matrices L,R encode an anisotropic (or correlated) source of randomness—a single element of the source matrix S_{ij} is perturbed by a noisy term $L_{ii}R_{jj}X_{ij}$, i.e., with variance $\sigma_{ij}^2 = (L_{ii}R_{jj})^2$. The absence of any structure means setting $S = 0$ and $L = R = 1$, which reduces Eq. (1) to standard RMT models of pure randomness.

There are at least two strategies for studying the model (1)—we look at either the eigenvalues or the singular values of M (equivalently at the eigenvalues of $M^\dagger M$). The first approach is limited to square matrices, whereas the second route is the main idea behind the principal component analysis in which, in general, rectangular data matrices M are investigated. In this work, we focus on the first approach and study the statistics of the eigenvalues. It is well known that the symmetries of M constrain the position of its eigenvalues. Here, however, we drop any symmetry constraints and focus on the case in which eigenvalues are spread over the whole complex plane. In what follows, we discuss a couple of instances that can be realized with the model (1) and which are interesting from a practical as well as from a theoretical perspective.

In finance, one studies the markets to make educated guesses about their future behavior, including the search for possibly profitable correlations. Toward that end, one typically considers N assets in T time slices that may be ordered in a rectangular $N \times T$ matrix M. We set $S = 0$ and interpret L,R as noise correlation matrices in both time and space. Because M is rectangular, the spectral density of $M^T M$ is studied, and thus we arrive at the doubly correlated Wishart model [5]. As a second example, in wireless telecommunication Eq. (1) arises in multiple-input–multiple-output (MIMO) systems as a complex $N_r \times N_t$ transmission matrix M between N_t transmitters and N_r receivers [6].

As a physics application, we consider a Hermitian Hamiltonian M that models an ensemble of charged spinless particles interacting with a strong external magnetic field [7]. In this instance, we set $S = e^{-\tau} H_0$, $LR = \sqrt{1 - e^{-2\tau}}$, and both H_0 and X are random matrices drawn from the Gaussian unitary ensemble (GUE). The parameter τ is proportional to the applied magnetic field. For moderate fields, a different random matrix model of (1) applies—a transition between a Gaussian orthogonal ensemble (GOE) and a GUE happens due to the breaking of time-reversal invariance. In this regime, we set $LR = i\alpha$ while the random matrices S and X are symmetric, $S = S^T$, and antisymmetric, $X = -X^T$, respectively. Even though we drop the positivity condition of L,R and consider a random matrix S, the model described is still of the form (1). As the parameter α that is proportional to the field varies between 0 and 1, a transition between GOE and GUE takes place.

Independently, the rich mathematical structure of models of the type (1) has attracted a lot of attention in its own right. These ensembles are known in the RMT community as "external source models". So far they were mostly considered for $L = R = 1$ and Hermitian X [8–11]. These models also have a natural interpretation in terms of Dyson's Brownian motion for the stochastic evolution in time τ, when we set $LR = \sqrt{\tau}$ and view S as the initial matrix [12,13].

Although all of the above examples contain either complex or real matrices M with a purely real spectrum, there are situations in which symmetry constraints are not present and the spectrum spreads over the whole complex plane. One of the

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main tenets of quantum mechanics for closed systems is the hermiticity of the Hamiltonian, while dropping it is an oft-used effective way to describe open systems, i.e., to account for the environment. As a consequence, complex energies of the type $E = \varepsilon - i\Gamma$ arise that correspond to resonant states. Such an energy eigenstate $|\phi_E(t)\rangle = e^{-iEt}|\phi_E(0)\rangle$ not only oscillates with a frequency ε, but it also decays with a characteristic time $1/\Gamma$. Random matrix models of this type were used for studying quantum chaotic scattering in open cavities [14]. In this case, the matrix S is drawn from the GUE, $LR = -i\pi$, and $X = W^\dagger W$ models a random interaction between the cavity and its surroundings, where W is drawn from a complex Girko-Ginibre ensemble.

As a second application of non-Hermitian matrices, we mention efforts to construct mathematical models of neuronal networks [15–17]. Here, M represents the neuronal adjacency matrix, and we begin by setting $S = 0, L = R = 1$. In this context, however, an additional constraint is needed—each matrix row must be either purely negative or purely positive, which reflects Dale's law of neuronal behavior. Moreover, a recent paper [18] argued that also the S, L, and R matrices in the model (1) might be of significance.

In the sequel, we consider matrices X drawn from the Girko-Ginibre ensemble (i.e., a matrix with complex Gaussians random entries) as well as various types of structural matrices S, L, and R. In Sec. II we compute an exact formula for the spectral density of M and arbitrary matrices S, L, and R. In Sec. III we investigate particular cases: a normal matrix S and arbitrary matrices L,R, a vanishing source $S = 0$ and trivial $L = R = 1$, and a rank-1 non-normal source S with $L = R = 1$. Eventually, we comment on the spectral formula for a related problem of eigenvalues of M^{-1}. We summarize and conclude in Sec. IV.

II. SPECTRAL DENSITY OF *M*

We now describe the model (1) in greater detail. Let X be an $N \times N$ matrix drawn from a complex Girko-Ginibre ensemble,

$$P(X)dX = C^{-1}\exp(-\mu \mathrm{Tr} X^\dagger X)dX, \tag{2}$$

where μ is an (inverse) variance parameter and $C = (\pi/\mu)^{N^2}$ is the normalization constant. The flat measure over the matrices X is denoted dX. All matrices S, L, and R are $N \times N$, with L,R being positive-definite and diagonal. The source matrix S is in the most general form given by $S = D + T$, where D is diagonal and T is strictly upper triangular. These reduced forms are not restrictive because the spectrum of M is unitarily invariant. In particular, the Schur decomposition of the source matrix reads $S = U^\dagger(D + T)U$ for a particular unitary matrix U. When $T = 0$, the source matrix is called normal, otherwise it is non-normal.

A basic statistical quantity characterizing the model (1) is the spectral density,

$$\rho(z,\bar{z}) = \frac{1}{N}\left\langle \sum_{i=1}^{N} \delta^{(2)}(z - m_i)\right\rangle_P, \tag{3}$$

depending on the complex variable z. The m_i are the eigenvalues of M. We use the two-dimensional Dirac δ function due to complexity of the spectrum; the average is taken over the random measure (2).

Many authors have studied the spectral density (3) in the large-N limit [19–21]. In particular, convenient quaternionic-hermitization methods [22,23] were developed to complete this task. For $L = R = 1$ and a general normal source S, spectral density in the large-N limit was found in Ref. [24], whereas the $L,R \neq 1$ generalization was recently studied in Ref. [18]. For finite matrix size, a formula for the spectral density was calculated in Ref. [25] for $L = R = 1$ and a normal source term S only. In this work, we address the cases $L,R \neq 1$ as well as non-normal S.

To find the spectral density, we introduce as a generating function the averaged ratio of determinants,

$$\mathcal{R}_{L,R}(Z,V) = \left\langle \frac{\det(Z - \mathcal{M})}{\det(V - \mathcal{M})}\right\rangle_P, \tag{4}$$

with the $2N \times 2N$ block matrices

$$\mathcal{M} = \begin{pmatrix} 0 & M \\ M^\dagger & 0 \end{pmatrix}, \tag{5}$$

$$Z = \begin{pmatrix} L^2 w & z\mathbf{1}_N \\ \bar{z}\mathbf{1}_N & -R^2\bar{w} \end{pmatrix}, \quad V = \begin{pmatrix} L^2 u & v\mathbf{1}_N \\ \bar{v}\mathbf{1}_N & -R^2\bar{u} \end{pmatrix}, \tag{6}$$

where $\mathbf{1}_N$ denotes the $N \times N$ unit matrix. We notice that the matrices Z and V depend on the complex variables z, u, v, and w. For $u = w = 0$, we recover the special case

$$\mathcal{R}_{L,R}(z,v) = \left\langle \frac{\det[(z - M)(\bar{z} - M^\dagger)]}{\det[(v - M)(\bar{v} - M^\dagger)]}\right\rangle_P. \tag{7}$$

Although the variables u,w have an interesting interpretation in terms of the eigenvectors [26], we only use their regulatory properties—as long as $u,w \neq 0$, the ratio is finite for all complex v. Importantly, the spectral density is generated by taking proper derivatives of the averaged ratio equation,

$$\rho(z,\bar{z}) = -\frac{1}{N\pi} \lim_{w,u \to 0} \frac{\partial}{\partial \bar{z}} \lim_{v \to z} \frac{\partial}{\partial v}\mathcal{R}_{L,R}(Z,V), \tag{8}$$

introduced in Ref. [27] for $L = R = 1$.

We use the supersymmetry method to calculate the generating function $\mathcal{R}_{L,R}$. Although the basic steps are by now standard, the details are rather involved in the present study. We refer the interested reader to Appendix A. We eventually arrive at the twofold integral representation

$$\mathcal{R}_{L,R} = \frac{4i}{\pi}\int_{-\infty}^{\infty} dg \int_0^{\infty} df\, S(f,g_-), \tag{9}$$

with $g_- = g - i\epsilon$ and the integrand

$$S(f,g_-) = e^{-\mu(g_-^2 + f^2 + |w|^2 - |u|^2)} I_0(2\mu f|w|)K_0(2i\mu|u|g_-) \times g_- f G[\gamma_1 + (\mu - \gamma_2)(\mu - \gamma_3) + \gamma_4], \tag{10}$$

depending on the modified Bessel functions I_0 and K_0 of the first and second type, respectively. The functions G and γ_i are equal to

$$G = \frac{\det(-f^2\mathbf{1}_N - \Gamma_z\Omega_z)}{\det(g_-^2\mathbf{1}_N - \Gamma_v\Omega_v)},$$

$$\gamma_1 = f^2 g_-^2 \mathrm{Tr}[\mathbf{P}_v\mathbf{Q}_z]\mathrm{Tr}[\mathbf{P}'_v\mathbf{Q}'_z],$$

$$\gamma_2 = \mathrm{Tr}[\Omega_z\Gamma_v\mathbf{P}_v\mathbf{Q}_z], \quad \gamma_3 = \mathrm{Tr}[\Omega_v\Gamma_z\mathbf{Q}_z\mathbf{P}_v],$$

$$\gamma_4 = f^2\mathrm{Tr}[\Omega_v\mathbf{Q}'_z\Gamma_v\mathbf{P}_v\mathbf{Q}_z\mathbf{P}_v] + g_-^2\mathrm{Tr}[\Omega_z\mathbf{P}'_v\Gamma_z\mathbf{Q}_z\mathbf{P}_v\mathbf{Q}_z],$$

where we defined

$$\Omega_x = R^{-2}(\bar{x}\mathbf{1}_N - S^\dagger), \qquad \Gamma_x = L^{-2}(x\mathbf{1}_N - S),$$

$$\mathbf{P}_v = (g_-^2\mathbf{1}_N - \Omega_v\Gamma_v)^{-1}, \quad \mathbf{P}'_v = (g_-^2\mathbf{1}_N - \Gamma_v\Omega_v)^{-1},$$

$$\mathbf{Q}_z = (-f^2\mathbf{1}_N - \Omega_z\Gamma_z)^{-1}, \quad \mathbf{Q}'_z = (-f^2\mathbf{1}_N - \Gamma_z\Omega_z)^{-1}.$$

III. PARTICULAR CASES

So far, the result (9) for the generating function is exact for any matrix dimension N and is valid for any structural matrices L, R, and S. Although the integrand (10) is, in general, rather complicated, the integral can be worked out explicitly for certain subclasses of L, R, and S. We are particularly interested in the following three cases:

(i) Vanishing source $S = 0$ and trivial $L = R = 1$.

(ii) Normal source S and variance matrices L,R arbitrary.

(iii) Non-normal source S of rank 1 and trivial variance matrices $L = R = 1$.

We compute these cases and discuss them in the sequel.

A. Vanishing source $S = 0$ and $L = R = 1$

We consider the case $S = 0$ and $L = R = 1$ in which a simple spectral density formula is known from the work of Ginibre [28]. The generating function (9) reads

$$\mathcal{R}_G = N(i_N j_N - i_{N-1}j_{N+1}) - \mu i_{N-1}j_{N+1}(\bar{v}z + \bar{z}v) + \mu(i_{N-1}j_{N+2}|v|^2 + i_{N-2}j_{N+1}|z|^2), \qquad (11)$$

where the building blocks i_m (fermionic type) and j_m (bosonic type) read

$$i_m = \frac{e^{-\mu|w|^2}}{m!}\int_0^\infty d\rho\, e^{-\rho} I_0(2\sqrt{\mu\rho}|w|)(\rho + \mu|z|^2)^m, \qquad (12)$$

$$j_m = \frac{(m-1)!}{2\pi i}\oint_\Gamma dp \sum_{k=0}^\infty \frac{U_{k+1,1}(\mu|u|^2)p^k}{(p+\mu|v|^2)^m}, \qquad (13)$$

where Γ is a contour encircling $-\mu|v|^2$ counterclockwise and $U_{a,b}(x) = U(a,b,x)$ is the Tricomi confluent hypergeometric function. The contour integral reformulation of the bosonic block j_m is derived in Appendix B.

Before proceeding, we cross-check the generating function (11) with similar calculations carried out for the chiral Gaussian unitary ensemble. Toward that end, we set $z = v = 0$, and the generating function reduces to

$$\mathcal{R}_{\mathrm{chGUE}} = \left\langle \frac{\det(|w|^2 + XX^\dagger)}{\det(|u|^2 + XX^\dagger)} \right\rangle_P. \qquad (14)$$

We also set $\mu = N$ and arrive at

$$\mathcal{R}_{\mathrm{chGUE}} = N(i_N j_N - i_{N-1}j_{N+1}).$$

By Eqs. (12) and (13) we find the fermionic and bosonic building blocks

$$i_m = L_m(-N|w|^2), \quad j_m = (m-1)!U_{m,1}(N|u|^2),$$

which reproduces the results of Ref. [29].

In the present study, we are interested in the complementary limit, i.e., we set $u,w \to 0$ and look at $z,v \neq 0$. Toward that end, we use the identity

$$k!U_{k+1,1}(\mu|u|^2) = e^{\mu|u|^2}\Gamma(0,\mu|u|^2)L_k(-\mu|u|^2) + \tilde{L}_k(-\mu|u|^2),$$

which when applied to the bosonic block (13) induces a splitting into two parts,

$$j_m(v,u) = j_m^{(\mathrm{sing})}(v,u) + j_m^{(\mathrm{reg})}(v,u). \qquad (15)$$

Because the ratio $\mathcal{R}_G$ is linear in j_m, we find the same type of separation $\mathcal{R}_G = \mathcal{R}_G^{(\mathrm{sing})} + \mathcal{R}_G^{(\mathrm{reg})}$. We apply the operator $\frac{\partial}{\partial\bar{z}}\lim_{v\to z}\frac{\partial}{\partial v}$ to this ratio, we find by symbolic calculation that $\frac{\partial}{\partial\bar{z}}\lim_{v\to z}\frac{\partial}{\partial v}\mathcal{R}_G^{(\mathrm{sing})} = 0$, and so the spectral density given by Eq. (8) simplifies

$$-N\pi\rho = \lim_{w,u\to 0}\frac{\partial}{\partial\bar{z}}\lim_{v\to z}\frac{\partial}{\partial v}\mathcal{R}_G = \lim_{w,u\to 0}\frac{\partial}{\partial\bar{z}}\lim_{v\to z}\frac{\partial}{\partial v}\mathcal{R}_G^{(\mathrm{reg})}. \qquad (16)$$

It depends, therefore, only on $j_m^{(\mathrm{reg})}(v,u)$. We exchange the limits in Eq. (16) and define the modified ratio $\tilde{\mathcal{R}}_G$:

$$\tilde{\mathcal{R}}_G = \lim_{w,u\to 0}\mathcal{R}_G^{(\mathrm{reg})} = \mathcal{R}_G[i_m \to \tilde{i}_m, j_m \to \tilde{j}_m], \qquad (17)$$

where the blocks are given by $\tilde{i}_m(z) = \lim_{w\to 0} i_m(z,w)$ and $\tilde{j}_m(v) = \lim_{u\to 0} j_m^{(\mathrm{reg})}(v,u)$. Although $\mathcal{R}_G \neq \tilde{\mathcal{R}}_G$, the spectral density $\rho = \tilde{\rho}$ agrees exactly,

$$\tilde{\rho}(z,\bar{z}) = -\frac{1}{N\pi}\frac{\partial}{\partial\bar{z}}\lim_{v\to z}\frac{\partial}{\partial v}\tilde{\mathcal{R}}_G. \qquad (18)$$

The modified building blocks read

$$\tilde{i}_m = \frac{1}{m!}\int_0^\infty d\rho\, e^{-\rho}(\rho + \mu|z|^2)^m, \qquad (19)$$

$$\tilde{j}_m = -\frac{(m-1)!}{2\pi i}\oint_\Gamma dp \frac{e^p[\gamma + \Gamma(0,p) + \ln p]}{(p+\mu|v|^2)^m}, \qquad (20)$$

where we used the identity

$$\sum_{m=0}^\infty \frac{1}{m!}\tilde{L}_m(0)p^m = -e^p[\gamma + \Gamma(0,p) + \ln p] \qquad (21)$$

for the modified Laguerre polynomials $\tilde{L}_m$ with γ denoting the Euler constant. This identity follows from the fact that $\tilde{L}_m(0) = -\sum_{k=1}^m \frac{1}{k}$ are the (negative) harmonic numbers.

Directly from Eqs. (19) and (20) we derive the iterative formulas

$$\tilde{i}_m = \tilde{i}_{m-1} + (\mu|z|^2)^m (m!)^{-1},$$
$$\tilde{j}_m = \tilde{j}_{m+1} - (\beta-1)!(\mu|v|^2)^{-m} e^{-\mu|v|^2}\tilde{i}_{m-1}(v),$$

and we use them to reexpress the generating function

$$\tilde{\mathcal{R}}_G = \mu \tilde{i}_{N-1}\tilde{j}_{N+1}|v-z|^2 + \frac{e^{-\mu|v|^2}}{|v|^{2N}}[\tilde{i}_{N-1}(z)|v|^{2N} - \tilde{i}_{N-1}(v)|z|^{2N}], \tag{22}$$

where we have written out explicitly the argument of $\tilde{i}$ to avoid confusion. At this point, we observe that the generating function vanishes for $z = v$, $\tilde{\mathcal{R}}_G = 0$. It is thus evident that the derivative operator of Eq. (18) only produces contributions due to the second term. Lastly, by using $\partial_{\bar{z}}\tilde{i}_m = \mu z \tilde{i}_{m-1}$ and $\partial_v \tilde{j}_m = -\mu \bar{v} \tilde{j}_{m+1}$, we recover the well-known formula

$$\rho_G = \frac{\mu}{N\pi} e^{-\mu|z|^2} \sum_{k=0}^{N-1} \frac{(\mu|z|^2)^k}{k!} \tag{23}$$

for the spectral density, which often appears for $\mu = N$.

B. Normal S and arbitrary L,R

In this case, all structure matrices L, R, and S are diagonal,

$$S = \mathrm{diag}(\underbrace{s_1,\ldots,s_1}_{u_1}, \underbrace{s_2,\ldots,s_2}_{u_2}, \underbrace{\ldots,s_x}_{\ldots u_x}),$$
$$L = \mathrm{diag}(\underbrace{l_1,\ldots,l_1}_{v_1}, \underbrace{l_2,\ldots,l_2}_{v_2}, \underbrace{\ldots,l_y}_{\ldots v_y}),$$
$$R = \mathrm{diag}(\underbrace{r_1,\ldots,r_1}_{w_1}, \underbrace{r_2,\ldots,r_2}_{w_2}, \underbrace{\ldots,r_z}_{\ldots w_z}),$$

with three sets of multiplicities u_i, v_i, w_i, which should not be confused with the above employed complex variables u, v, w. Here, x, y, z are the numbers of different entries in the structure matrices L, R, and S, respectively, thereby defining the sizes of the sets. The multiplicities in each set add up to N. Because the integrand (10) only depends on the products $(\Omega_x)_{ii}(\Gamma_y)_{ii}$, we introduce a structured source matrix of the form

$$\alpha_{xy} = \Omega_x\Gamma_y = (LR)^{-2}(\bar{x}\mathbf{1}_N - S^\dagger)(y\mathbf{1}_N - S), \tag{24}$$

which depends on all three matrices L, R, and S. It is accompanied by a merged multiplicity vector $\vec{n}$. We define it by the following construction: we first form the multiplicity vectors $\vec{u} = (u_1,\ldots,u_x)$, $\vec{v} = (v_1,\ldots,v_y)$, and $\vec{w} = (w_1,\ldots,w_z)$ corresponding to the matrices S, L, and R, respectively. The vector $\vec{u}$ is graphically represented by a column of N points, which are ordered in x groups according to the multiplicities u_i. The points within each of these x groups are given the same (arbitrary) color, which is only used to distinguish the different groups. We refer to the first and last points in each group as the boundary. The vectors $\vec{v}, \vec{w}$ are represented accordingly. The multiplicity vector $\vec{n} = (n_1,\ldots,n_k)$ is then constructed as a vector that has a boundary whenever *at least* one of the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ has one. We illustrate this by the example in Fig. 1 in which the vector $\vec{u}$ is represented by

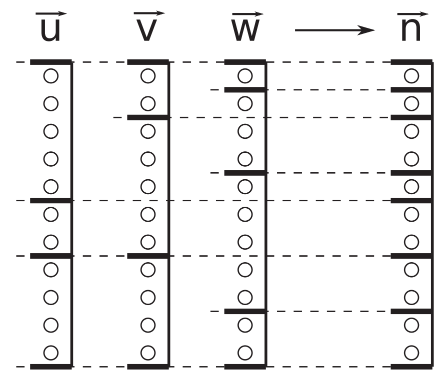

FIG. 1. Construction of the multiplicity vector $\vec{n} = (1,1,2,1,2,2,2)$ from $\vec{u} = (5,2,4)$, $\vec{v} = (2,5,4)$, and $\vec{w} = (1,3,5,2)$. The points depict groups of sizes determined by the corresponding multiplicities. Horizontal lines (both solid and dashed) are drawn along the boundaries of the groups of any of the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$, visualizing the construction of the merged vector $\vec{n}$.

$N = 11$ points ordered in $x = 3$ groups with multiplicities $u_1 = 5, u_2 = 2$, and $u_3 = 4$ with $5+2+4 = 11$. As seen, the multiplicities for the other two vectors differ. We juxtapose the point sets of all three multiplicity vectors along with the constructed $\vec{n}$. From now on, we only use the merged vector $\vec{n}$. We introduce the dimension $d(\vec{n})$ of the vector $\vec{n}$ as the number of differing groups, e.g., $d(\vec{n}) = 7$ in the above example. We also introduce the length $|\vec{n}| = \sum_{i=1}^{d(\vec{n})} n_i$. The generating function (9) can then be cast into the form

$$\frac{1}{C}\mathcal{R}_{L,R} = i_{\vec{n}}j_{\vec{n}} - \sum_{i=1}^{d(\vec{n})}\frac{\mu}{n_i}\left(\alpha_{zv}^i + \alpha_{vz}^i + \frac{N}{\mu}\right) i_{\vec{n}-\vec{e}_i}j_{\vec{n}+\vec{e}_i} + \sum_{i,j=1}^{d(\vec{n})}\frac{\mu^2\alpha_{zv}^i}{n_i n_j}\left[\left(\alpha_{vz}^j - \alpha_{vz}^i\right) i_{\vec{n}-\vec{e}_i-\vec{e}_j}j_{\vec{n}+\vec{e}_i+\vec{e}_j}\right] + \sum_{i,j=1}^{d(\vec{n})}\frac{\mu}{n_j}\left[\alpha_{vv}^i i_{\vec{n}-\vec{e}_j}j_{\vec{n}+\vec{e}_i+\vec{e}_j} + \alpha_{zz}^i i_{\vec{n}-\vec{e}_i-\vec{e}_j}j_{\vec{n}+\vec{e}_j}\right], \tag{25}$$

where α_{xy}^i is the ith element of the diagonal matrix (24), $C = \prod_{i=1}^{d(\vec{n})} n_i$, and the $\vec{e}_i$'s are k-dimensional unit vectors in the ith direction. These vectors $\vec{e}_i$ are used to conveniently add or subtract a single source from the vector $\vec{n}$. Similarly to the Ginibre case considered in Sec. III A, the result (25) contains fermionic and bosonic building blocks. The former is given by

$$i_{\vec{m}} = \frac{e^{-\mu|w|^2}}{\prod_{i=1}^{d(\vec{m})} m_i!}\int_0^\infty d\rho\, e^{-\rho} I_0(2\sqrt{\mu\rho}|w|)\prod_{i=1}^{d(\vec{m})}\left(\rho + \mu\alpha_{zz}^i\right)^{m_i}, \tag{26}$$

where we set $i_{\vec{m}} = 0$ if some element of the multiplicity vector $\vec{m}$ is negative. The bosonic counterpart reads

$$j_{\vec{m}}(v,u) = \frac{2i\mu}{\pi} \prod_{i=1}^{d(\vec{m})} \frac{(m_i-1)!}{(-\mu)^{m_i}} e^{\mu|u|^2} \times \int_{-\infty}^{\infty} dg\, g_- e^{-\mu g_-^2} K_0(2i\mu|u|g_-) \times \prod_{i=1}^{d(\vec{m})} \left(g_-^2 - \alpha_{vv}^i\right)^{-m_i}. \quad (27)$$

Analogously to formula (13), we notice that the bosonic building block is expressed as the contour integral

$$j_{\vec{m}}(v,u) = \frac{\prod_{i=1}^{d(\vec{m})}(m_i-1)!}{2\pi i} \oint_{\Gamma_s} dp \sum_{k=0}^{\infty} \frac{U_{k+1,1}(\mu|u|^2)p^k}{\prod_{i=1}^{d(\vec{m})}\left(p+\mu\alpha_{vv}^i\right)^{m_i}}, \quad (28)$$

where the contour Γ_s encircles all sources $-\mu\alpha_{vv}^i$ counterclockwise. Details of the calculation are provided in Appendix B.

We follow the discussion in Sec. III A to address the limit $u,w \to 0$. As the procedure is analogous, we can immediately write down the modified generating function

$$\tilde{\mathcal{R}}_{L,R} = \mathcal{R}_{L,R}[i_{\vec{m}}(z,w) \to \tilde{i}_{\vec{m}}(z), j_{\vec{m}}(v,u) \to \tilde{j}_{\vec{m}}(v)], \quad (29)$$

with new building blocks $\tilde{i}_{\vec{m}}(z) = \lim_{w\to 0} i_{\vec{m}}(z,w)$ and $\tilde{j}_{\vec{m}}(v) = \lim_{u\to 0} j_{\vec{m}}^{(\mathrm{reg})}(v,u)$. We stress that this procedure is not an approximation—although we have $\tilde{\mathcal{R}}_{L,R} \neq \mathcal{R}_{L,R}$, the spectral densities obtained by Eqs. (8) and (18) agree exactly $\tilde{\rho} = \rho$. The modified building blocks are given by

$$\tilde{i}_{\vec{m}} = \frac{1}{\prod_{i=1}^{d(\vec{m})} m_i!} \int_0^{\infty} d\rho\, e^{-\rho} \prod_{i=1}^{d(\vec{m})} \left(\rho + \mu\alpha_{zz}^i\right)^{m_i},$$
$$\tilde{j}_{\vec{m}} = -\frac{\prod_{i=1}^{d(\vec{m})}(m_i-1)!}{2\pi i} \oint_{\Gamma_s} dp \frac{e^p[\gamma + \Gamma(0,p) + \ln p]}{\prod_{i=1}^{d(\vec{m})}\left(p+\mu\alpha_{vv}^i\right)^{m_i}}, \quad (30)$$

where we used the identity (21).

The final formula for the spectral density in the case of a normal source S and nontrivial L,R then reads

$$\tilde{\rho} = -\frac{1}{N\pi}\frac{\partial}{\partial \bar{z}} \lim_{V \to Z} \frac{\partial}{\partial v} \tilde{\mathcal{R}}_{L,R}(z,v), \quad (31)$$

together with the definitions (25), (29), and (30). We demonstrate the utility of our analytical result in Fig. 2 by comparing it with numerical simulations. Adding (structured) noise LXR produces an overall eigenvalue spreading with anisotropic features reflecting the L,R covariance matrices. The density is concentrated around the initial eigenvalues of S and varies smoothly as we change the noise level μ, i.e., the inverse variance of the ensemble (2).

C. Non-normal rank-1 S and $L = R = 1$

A major reason to study models of the type (1) is the issue of spectral stability. How far do the eigenvalues of $S + Y$ spread around the eigenvalues of S for a small perturbation Y? This is especially interesting for finite rank sources S where

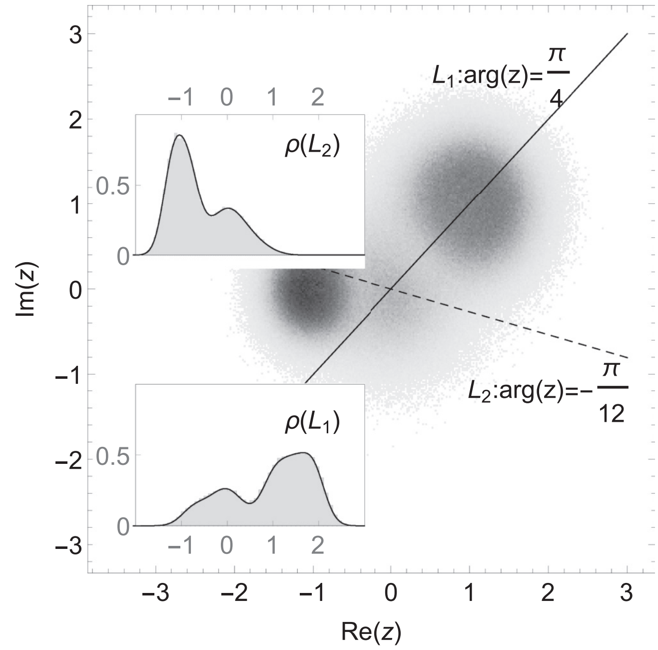

FIG. 2. Spectral density according to Eq. (31) as insets along two lines L_1 and L_2 in the complex plane, together with numerical simulations. The structural matrices are $S = \mathrm{diag}(-1,0,1+i)$, $L = \mathrm{diag}(3/4,1)$, and $R = \mathrm{diag}(1,5/4,1)$ with multiplicity vectors of $\vec{u} = (2,1,3)$, $\vec{v} = (2,4)$, and $\vec{w} = (2,1,3)$.

extremal (or outlier) eigenvalues emerge from the eigenvalue sea of the matrix Y. This phenomenon was studied in a Hermitian [30–32] as well as a non-Hermitian [33–35] setting. Here, we examine how the normal or non-normal character of the source influences the eigenvalue distribution. We consider a rank-1 source of the form

$$S = \alpha|n\rangle\langle m| \quad (32)$$

for complex parameter α and bras (kets) $\langle m|$ ($|n\rangle$) denoting the canonical matrix basis—the source matrix S has one nonzero element α placed on the off-diagonal. For the sake of simplicity, we choose the trivial variance structure $L = R = 1$. After a fair amount of algebra, we find the result

$$\mathcal{R}_{\mathrm{NN}} = R_0 + |\alpha|^2 R_1 + |\alpha|^4 R_2 + |\alpha|^6 R_3 + |\alpha|^8 R_4 \quad (33)$$

for the generating function. The formulas for the R_i's are lengthy and are thus shifted into Appendix C. Although the terms in Eq. (33) turn out to lack structure, they are still assembled from the bosonic and fermionic building blocks similar to Eq. (26),

$$i_{k,l}(z,w) = \frac{(-1)^k}{\mu^{k+2l+1}} e^{-\mu|w|^2} \int_0^{\infty} d\rho\, e^{-\rho} I_0(2\sqrt{\mu\rho}|w|) \times (\rho + \mu|z|^2)^k (\rho + \mu k_z^+)^l (\rho + \mu k_z^-)^l, \quad (34)$$

and Eq. (27),

$$j_{q,r}(v,u) = \frac{2}{i\pi} e^{\mu|u|^2} \int_{-\infty}^{\infty} dg\, g_- e^{-\mu g_-^2} K_0(2i\mu|u|g_-) \times (g_-^2 - |v|^2)^{-q} (g_-^2 - k_v^+)^{-r} (g_-^2 - k_v^-)^{-r}, \quad (35)$$

where $k_x^{\pm} = \frac{1}{2}(|\alpha|^2 + 2|x|^2 \pm |\alpha|\sqrt{4|x|^2 + |\alpha|^2})$. By investigating the terms in each of the R_i's, we find the conditions $l = -1,0,1$, $k \geqslant 0$ and $q + r \geqslant 1$, $r = 1,2,3$, for the indices of $i_{k,l}$ and $j_{q,r}$, respectively. We employ the same steps as in Sec. III B, obtain the generating function $\tilde{\mathcal{R}}_{\rm NN}$, and construct the modified fermionic block,

$$\tilde{i}_{k,0} = \frac{(-1)^k k!}{\mu^{k+1}} (\tilde{i}_G)_k, \tag{36}$$

$$\tilde{i}_{k,1} = \tilde{i}_{k+2,0} - |\alpha|^2 (\tilde{i}_{k+1,0} + |z|^2 \tilde{i}_{k,0}), \tag{37}$$

$$\tilde{i}_{k,-1} = \frac{(-1)^k k!}{(k_z^+ - k_z^-)\mu^k} \sum_{l=0}^{k} \frac{(\mu|z|^2)^l}{l!} \times [U_{1,1+l-k}(\mu k_z^-) - U_{1,1+l-k}(\mu k_z^+)], \tag{38}$$

where $\tilde{i}_G$ is the Ginibre block of Eq. (19) and $k \geqslant 0$. We relegate the derivation of Eq. (38) to Appendix B. The bosonic block reads

$$\tilde{j}_{q,r} = -\frac{(-\mu)^{q+2r-1}}{2\pi i} \times \oint_\Gamma \frac{dp\, e^p \ln p}{(p + \mu|v|^2)^q (p + \mu k_v^-)^r (p + \mu k_v^+)^r}, \tag{39}$$

where $q \geqslant 0$, $r \geqslant 1$, and the contour Γ encircles both $-\mu|v|^2$ and $-\mu k_v^{\pm}$. Lastly, we obtain the formulas for $q = -1,-2$,

$$\tilde{j}_{-1,2} = \tfrac{1}{2}(\tilde{j}_{0,2_-} + \tilde{j}_{0,2_+} + |\alpha|^2 \tilde{j}_{0,2}), \tag{40}$$

$$\tilde{j}_{-1,3} = \tfrac{1}{2}(\tilde{j}_{0,3_-} + \tilde{j}_{0,3_+} + |\alpha|^2 \tilde{j}_{0,3}), \tag{41}$$

$$\tilde{j}_{-2,3} = \tfrac{1}{4}[\tilde{j}_{0,3_{--}} + 2\tilde{j}_{0,3_{+-}} + \tilde{j}_{0,3_{++}} + |\alpha|^4 \tilde{j}_{0,3} + 2|\alpha|^2(\tilde{j}_{0,3_+} + \tilde{j}_{0,3_-})], \tag{42}$$

where the subscripts $\pm$ indicate that the underlying multiplicity vector $\vec{x} = (q, r-1, r)$ is applied with decrement to the source at $nk_v^{\pm}$.

Finally, we obtain the spectral density (3) analytically and plot it in Fig. 3. To facilitate a comparison, we juxtapose it with the analogous results for the case of a rank-1 normal source S and for the Ginibre case (23). A non-normal source S (third row in Fig. 3) does not produce, on average, outlier eigenvalues in the spectrum, in contrast to the normal source S (second row in Fig. 3), where we find an island around $\alpha = 10$. Instead, in the non-normal case we observe something like a blowup of the spectral bulk. The first row in Fig. 3 is devoted to the case of a vanishing source, $S = 0$. Near $z = 0$, both the normal and the vanishing source produce similarly shaped spectral densities—the only difference between these cases is the presence or absence of the finite-rank island.

D. Spectrum of M^{-1}

As a last application, we discuss how to infer somewhat gratuitously the spectrum of $(S + X)^{-1}$ from the results for the spectrum of $S + X$. For simplicity, we deal with a normal source S only and set $L = R = 1$. Toward that end, we define

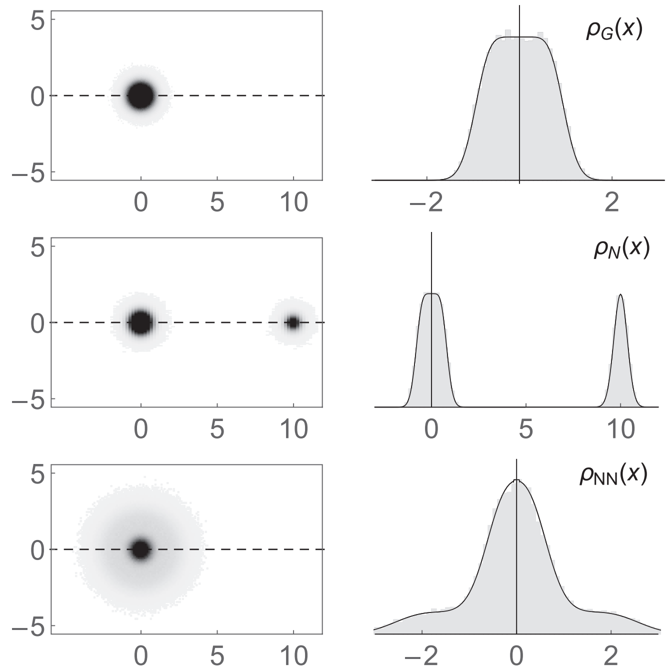

FIG. 3. Left-hand side: complex plane of eigenvalues, from top to bottom for unperturbed $S = 0$ (Ginibre), normal perturbation $S = 10|1\rangle\langle 1|$, and non-normal perturbation $S = 10|2\rangle\langle 1|$. Right-hand side: numerical simulations and analytical results for the spectral densities ρ_G, ρ_N, and $\rho_{\rm NN}$ along the real axis line (dashed lines on the left-hand side). Numerical simulations are for matrices of size $N = 4$, $\alpha = 10$, and we set $\mu = N$.

a generating function $\mathcal{R}_{-1}$ for the inverse as

$$\mathcal{R}_{-1}(Z,V) = \left\langle \frac{\det(Z - \mathcal{M}_{-1})}{\det(V - \mathcal{M}_{-1})} \right\rangle_P = \frac{\det Z}{\det V} \mathcal{R}_{1,1}(Z', V'), \tag{43}$$

and we relate it to the generating function (4) previously considered. The matrices $\mathcal{M}_{-1} = \begin{pmatrix} 0 & M^{-1} \\ M^{\dagger,-1} & 0 \end{pmatrix}$ and Z', V' are rearranged versions of the inverse matrices Z^{-1}, V^{-1} of Eq. (6),

$$X' = \begin{pmatrix} (X^{-1})_{22} & (X^{-1})_{21} \\ (X^{-1})_{12} & (X^{-1})_{11} \end{pmatrix}, \quad X = Z, V. \tag{44}$$

We thus conclude that the whole calculation discussed in Sec. III B can be repeated with only making the replacements $w \to -w G_{zw}$, $z \to \bar{z} G_{zw}$, $u \to -u G_{vu}$, and $v \to \bar{v} G_{vu}$ with $G_{xy} = (|x|^2 + |y|^2)^{-1}$. We again conduct the $\mathcal{R} \to \tilde{\mathcal{R}}$ procedure and eventually find that only the source matrix of Eq. (24) is modified according to

$$\alpha_{xy} \to (\alpha^{-1})_{xy} = \alpha_{x^{-1}y^{-1}} = (\bar{x}^{-1}\mathbf{1}_N - S^\dagger)(y^{-1}\mathbf{1}_N - S).$$

The modified ratio for the problem of finding the spectrum of $(S + X)^{-1}$ reads

$$\tilde{R}_{-1} = \left(\frac{|z|^2}{|v|^2}\right)^{|\vec{n}|} \tilde{\mathcal{R}}_{1,1}[\alpha_{xy} \to (\alpha^{-1})_{xy}], \tag{45}$$

where the generating function $\tilde{\mathcal{R}}_{1,1}$ is that of Eq. (29) and the constituent fermionic and bosonic blocks (30) are affected accordingly. In particular, we calculate the spectral density for

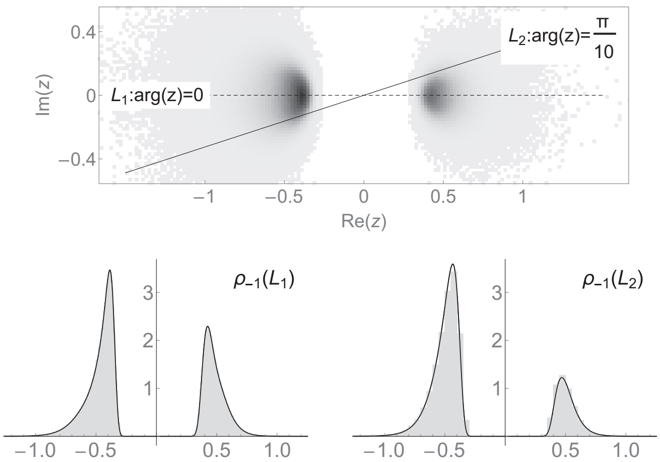

FIG. 4. A numerical simulation (top histogram) along with analytic spectral density plots of matrix $(S+X)^{-1}$ along two straight lines L_1 (bottom-left plot) and L_2 (bottom-right plot) for an external source setup of $S=(-2,2)$ with multiplicities $\vec{n}=(4,2)$.

an inverse matrix X^{-1} as

$$\rho_{G,-1}=\frac{\mu e^{-\frac{\mu}{|z|^2}}}{N\pi|z|^4}\sum_{k=0}^{N-1}\frac{1}{(k)!}\left(\frac{\mu}{|z|^2}\right)^k, \tag{46}$$

obtained from Eq. (22). This formula was also found in a recent work on the product of matrices [36].

In Fig. 4, the spectral density of $(S+X)^{-1}$ is depicted as calculated from the generating function (45) for the nonzero external source S.

IV. CONCLUSIONS

We have calculated exact spectral densities for a class of complex random matrix models of the form $M=S+LXR$ consisting of a noise part X and structure parts S,L,R. We found twofold integral formulas for arbitrary structural matrices. In greater detail, we investigated the case of a normal source matrix S and arbitrary diagonal matrices L,R, which are of particular interest. The resulting formulas are of a remarkably succinct form. We confirmed our analytical results by numerical simulations.

We showed how the presence or absence of the normality condition for S leads to a qualitatively different behavior of the eigenvalue densities. Our study was focused mainly on the finite rank source matrices where analytical solutions proved tractable. For a non-normal source, the most interesting feature is the lack of outliers, i.e., extreme values in the averaged spectral density. However, when imposing the normality condition on the source matrix S, the outliers are clearly present in the spectral density. Lastly, we looked at the problem of finding spectra of an inverse matrix M^{-1}, which, by using the approach in this paper, proved to be trivially connected to the spectrum of M.

Among the open problems in the context of our study, the question remains as to whether the normal versus non-normal dichotomy has any counterpart relevant for applications. Secondly, the information on eigenvectors is encoded in the objects of study, but due to the approach taken, it was completely omitted in our present work. Thirdly, issues related to universality seem feasible within our approach and are certainly worth future investigation.

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APPENDIX A: SUPERSYMMETRIC METHOD

We start off from the joint PDF (2):

$$P(X)dX=C^{-1}\exp(-\mu\mathrm{Tr}X^\dagger X)dX. \tag{A1}$$

As a first step we make the change of variables $Y=LXR$ implying $M=S+Y$ as well as $\mathcal{M}=\mathcal{S}+\mathcal{Y}$ defined by Eq. (5). The measure $P(X)dX$ now reads

$$P_{L,R}(Y)dY=C_{L,R}^{-1}\exp\left(-\mu\mathrm{Tr}R^{-2}Y^\dagger L^{-2}Y\right)dY, \tag{A2}$$

where the normalization constant is given as $C_{L,R}=(\pi/\mu)^{N^2}\det(LR)^2$. We open the ratio of determinants with the help of complex Grassmann variables χ_i and complex ordinary variables ϕ_i,

$$\frac{\det(Z-\mathcal{M})}{\det(V-\mathcal{M})}=c\int d[\phi,\chi]e^{iq^\dagger\mathrm{diag}(V-\mathcal{M},Z-\mathcal{M})q}, \tag{A3}$$

with a proper normalization constant c. We introduced the supervector $q=(\phi_1\phi_2\chi_1\chi_2)^T$ and the joint measure $d[\phi,\chi]=\prod_{i=1}^N d(\phi_1)_i d(\phi_2)_i d(\chi_1)_i d(\chi_2)_i$. Averaging with the distribution $P_{L,R}$ only affects the exponential terms proportional to Y, which are given by

$$\begin{aligned}e^{-iq^\dagger\mathrm{diag}(\mathcal{Y},\mathcal{Y})q}&=e^{-i(\phi_1^\dagger Y\phi_2+\chi_1^\dagger Y\chi_2+\phi_2^\dagger Y^\dagger\phi_1+\chi_2^\dagger Y^\dagger\chi_1)}\\&=e^{-i\mathrm{Tr}(E_1Y+E_2Y^\dagger)},\end{aligned}$$

where we set $(E_1)_{ij}=(\phi_2)_i(\bar{\phi}_1)_j-(\chi_2)_i(\bar{\chi}_1)_j$ and $(E_2)_{ij}=(\phi_1)_i(\bar{\phi}_2)_j-(\chi_1)_i(\bar{\chi}_2)_j$. The average is easily found to be

$$\int dY\,P_{L,R}(Y)e^{-i\mathrm{Tr}(E_1Y+E_2Y^\dagger)}=e^{-\frac{1}{\mu}\mathrm{Tr}E_1L^2E_2R^2}. \tag{A4}$$

To proceed further, we carry out a Hubbard-Stratonovich transformation

$$e^{-\frac{1}{\mu}\mathrm{Tr}E_1L^2E_2R^2}=c_0\int[d\Sigma]e^{-\mu F-q^\dagger Qq}, \tag{A5}$$

which reduces the fourth-order supervector terms to second order. The supermatrix Q appearing in the exponent is given by

$$Q=\begin{pmatrix}\mathcal{L}\mathrm{diag}(\sigma\mathbf{1}_N,-\bar{\sigma}\mathbf{1}_N) & \mathcal{L}\mathrm{diag}(\delta\mathbf{1}_N,\beta\mathbf{1}_N)\\ \mathcal{L}\mathrm{diag}(\bar{\delta}\mathbf{1}_N,\bar{\beta}\mathbf{1}_N) & \mathcal{L}\mathrm{diag}(\bar{\rho}\mathbf{1}_N,\rho\mathbf{1}_N)\end{pmatrix}, \tag{A6}$$

with $\mathcal{L}=\mathrm{diag}(L^2,R^2)$. It depends on four new complex integration variables, two ordinary ones σ and ρ as well as two anticommuting ones δ and β. The corresponding measure

$$d[\Sigma] = d^2\sigma d^2\rho d^2\delta d^2\beta \tag{A7}$$

is flat. We use the notation $d^2\delta = d\delta d\bar{\delta}$. The normalization constant in Eq. (A5) is given by $c_0 = \pi^{-2}$. The function $F = |\sigma|^2 + |\rho|^2 + \bar{\delta}\beta + \bar{\beta}\delta$ in the exponent yields the Gaussians needed to bring the supervector q to second order.

Thus, we can cast the generating function $\mathcal{R}_{L,R}$ into the form

$$\mathcal{R}_{L,R} = cc_0 \int d[\phi,\chi] \int d[\Sigma] e^{-\mu F + i q^\dagger A q}, \tag{A8}$$

where we introduced the supermatrix

$$A = \text{diag}(V - \mathcal{S}, Z - \mathcal{S}) + iQ. \tag{A9}$$

In the next step, we interchange the order of integration $d[\phi,\chi] \leftrightarrow d[\Sigma]$. This, however, has a subtle flaw: the resulting integral is no longer convergent in the ρ and σ directions, an issue addressed previously [37,38]. It is a problem arising in the supersymmetric method when the parametrization of the $d[\Sigma]$ manifold does not represent the symmetries of the $d[\phi,\chi]$ manifold. To fix it, we inspect the symmetries of the latter manifold and the transformations induced on the former manifold so that the overall ratio stays invariant. This results in the change of variables

$$\rho = \rho_1 + i\rho_2, \qquad \sigma = \sigma_1 + i\sigma_2,$$
$$\rho_1 = i\frac{w - \bar{w}}{2} + f\cos\phi, \qquad \rho_2 = -\frac{w + \bar{w}}{2} + f\sin\phi,$$
$$\sigma_1 = i\frac{u + \bar{u}}{2} - ig_-\sinh\gamma, \qquad \sigma_2 = \frac{u - \bar{u}}{2} + g_-\cosh\gamma.$$

We apply it before swapping the order of integration. Here, we introduced real commuting variables f, g, γ, and ϕ as well as a small imaginary increment, $g_- = g - i\epsilon$, with $\epsilon > 0$. The range of integration is $f \geqslant 0, \phi \in (0,2\pi], g \in \mathbb{R}, \gamma \in \mathbb{R}$. The anticommuting variables δ,β remain unchanged. The integral then becomes

$$\int d[\Sigma] e^{-\mu F + i q^\dagger A q} = \int d[\Sigma'](-ig_- f) e^{-\mu F' + i q^\dagger A' q}, \tag{A10}$$

with $d[\Sigma'] = df d\phi dg d\gamma d^2\delta d^2\beta$ and

$$F' = g_-^2 + f^2 + |w|^2 - |u|^2 + g_-(ue^\gamma - \bar{u}e^{-\gamma}) + if(we^{i\phi} - \bar{w}e^{-i\phi}) + \bar{\delta}\beta + \bar{\beta}\delta. \tag{A11}$$

We also introduced the transformed supermatrix

$$A' = \begin{pmatrix} A'_{BB} & A'_{BF} \\ A'_{FB} & A'_{FF} \end{pmatrix}, \tag{A12}$$

with the $2N \times 2N$ blocks

$$A'_{BB} = \begin{pmatrix} -L^2\sigma_- e^{-s} & v\mathbf{1}_N - S \\ \bar{v}\mathbf{1}_N - S^\dagger & -R^2\sigma_- e^{s} \end{pmatrix}, \quad A'_{BF} = \begin{pmatrix} i\delta L^2 & 0 \\ 0 & i\beta R^2 \end{pmatrix},$$
$$A'_{FF} = \begin{pmatrix} iL^2\rho e^{-i\phi} & z\mathbf{1}_N - S \\ \bar{z}\mathbf{1}_N - S^\dagger & iR^2\rho e^{i\phi} \end{pmatrix}, \quad A'_{FB} = \begin{pmatrix} i\bar{\delta}L^2 & 0 \\ 0 & i\bar{\beta}R^2 \end{pmatrix}.$$

After this change of variables, we may now safely interchange the order of integration and arrive at

$$\mathcal{R}_{L,R} = -ic_0 \int d[\Sigma'] g_- f e^{-\mu F'} \text{sdet}^{-1} A', \tag{A13}$$

where the integral over the supervector yielded the superdeterminant as an extension of Eq. (A3),

$$c \int d[\phi,\chi] e^{i q^\dagger A' q} = \text{sdet}^{-1} A'. \tag{A14}$$

The superdeterminant is known to satisfy the formula

$$\text{sdet}^{-x} A' = \frac{\det^x A'_{FF}}{\det^x A'_{BB}} \left(1 + x\text{Tr}A_0 + \frac{x}{2}\text{Tr}A_0^2 + \frac{x^2}{2}(\text{Tr}A_0)^2\right),$$

where $A_0 = {A'_{BB}}^{-1} A'_{BF} {A'_{FF}}^{-1} A'_{FB}$ for any integer x. This result enables us to integrate over the Grassmann variables δ,β in Eq. (A13). The integral

$$I(f,g,\phi,\gamma) = \int d^2\delta d^2\beta e^{-\mu(\bar{\delta}\beta + \bar{\beta}\delta)} \text{sdet}^{-1} A' \tag{A15}$$

can be written in the form

$$I = -G[\gamma_1 + (\mu - \gamma_2)(\mu - \gamma_3) + \gamma_4] \tag{A16}$$

after some algebra and by utilizing the standard normalization of the Berezin integrals to one. The individual terms are

$$G = \frac{\det(-f^2\mathbf{1}_N - \Gamma_z\Omega_z)}{\det(g_-^2\mathbf{1}_N - \Gamma_v\Omega_v)},$$
$$\gamma_1 = f^2 g_-^2 \text{Tr}[\mathbf{P}_v\mathbf{Q}_z]\text{Tr}[\mathbf{P}'_v\mathbf{Q}'_z],$$
$$\gamma_2 = \text{Tr}[\Omega_z\Gamma_v\mathbf{P}_v\mathbf{Q}_z], \quad \gamma_3 = \text{Tr}[\Omega_v\Gamma_z\mathbf{Q}_z\mathbf{P}_v],$$
$$\gamma_4 = f^2\text{Tr}[\Omega_v\mathbf{Q}'_z\Gamma_v\mathbf{P}_v\mathbf{Q}_z\mathbf{P}_v] + g_-^2\text{Tr}[\Omega_z\mathbf{P}'_v\Gamma_z\mathbf{Q}_z\mathbf{P}_v\mathbf{Q}_z],$$

where we defined

$$\Omega_x = R^{-2}(\bar{x}\mathbf{1}_N - S^\dagger), \qquad \Gamma_x = L^{-2}(x\mathbf{1}_N - S),$$
$$\mathbf{P}_v = (g_-^2\mathbf{1}_N - \Omega_v\Gamma_v)^{-1}, \quad \mathbf{P}'_v = (g_-^2\mathbf{1}_N - \Gamma_v\Omega_v)^{-1},$$
$$\mathbf{Q}_z = (-f^2\mathbf{1}_N - \Omega_z\Gamma_z)^{-1}, \quad \mathbf{Q}'_z = (-f^2\mathbf{1}_N - \Gamma_z\Omega_z)^{-1}.$$

At this point, we make the remarkable observation that the function I is independent of the variables γ and ϕ such that $I(f,g,\phi,\gamma) = I(f,g)$. Hence integrating over the fermionic variables effectively restores a certain invariance. We collect Eqs. (A13) and (A16) to obtain the generating ratio

$$\mathcal{R}_{L,R} = \frac{4i}{\pi} \int_{-\infty}^{\infty} dg \int_0^\infty df\, S(f,g_-), \tag{A17}$$

with the integrand

$$S(f,g_-) = e^{-\mu(g_-^2 + f^2 + |w|^2 - |u|^2)} I_0(2\mu f|w|) K_0(2i\mu|u|g_-) \times g_- f G[\gamma_1 + (\mu - \gamma_2)(\mu - \gamma_3) + \gamma_4], \tag{A18}$$

where I_0, K_0 are the Bessel functions of the first and second kind, respectively. They result from the following integrals over the γ,ϕ variables:

$$\int_{-\infty}^{\infty} d\gamma\, e^{-\mu g_-(ue^{\gamma}-\bar{u}e^{-\gamma})} = 2K_0(2i\mu|u|g_-),$$

$$\int_0^{2\pi} d\phi\, e^{-i\mu f(we^{i\phi}-\bar{w}e^{-i\phi})} = 2\pi I_0(2\mu f|w|),$$

where we set $u=|u|e^{i\theta}$, $w=|w|e^{i\psi}$, and we choose the argument of u to be $\theta=\pi/2$ to make the γ integral convergent. The angle of w is arbitrary since the ϕ integral is periodic, and so we can set $\psi=0$.

APPENDIX B: DERIVATION OF (28)

We start from Eq. (27):

$$j_{\vec{m}}(v,u) = \frac{2i\mu}{\pi}\prod_{i=1}^{k}\frac{(m_i-1)!}{(-\mu)^{m_i}}e^{\mu|u|^2}J_{\vec{m}}(v,u), \quad \text{(B1)}$$

$$J_{\vec{m}}(v,u) = \int_{-\infty}^{\infty} dg\, g_- e^{-\mu g_-^2}K_0(2i\mu|u|g_-)\prod_{i=1}^{k}\left(g_-^2-\alpha_{vv}^i\right)^{-m_i}, \quad \text{(B2)}$$

where we set $d(\vec{m})=k$ for brevity. Using the Lagrange interpolation formula, we find

$$\prod_{i=1}^{k}\left(g_-^2-\alpha_{vv}^i\right)^{-m_i} = \lim_{\gamma_1,\ldots,\gamma_k\to 1}\sum_{l=1}^{k}\mathcal{D}_l\left(g_-^2-\gamma_l\alpha_{vv}^l\right)^{-1},$$

with the operator $\mathcal{D}_l$ defined as

$$\mathcal{D}_l = \prod_{i=1}^{k}\frac{\left(\alpha_{vv}^i\right)^{1-m_i}}{(m_i-1)!}\frac{d^{m_i-1}}{d\gamma_i^{m_i-1}}\prod_{j=1(\neq l)}^{k}\left(\gamma_l\alpha_{vv}^l-\gamma_j\alpha_{vv}^j\right)^{-1},$$

so that the whole integral $J_{\vec{m}}$ is expressed as

$$J_{\vec{m}} = \lim_{\gamma_1,\ldots,\gamma_k\to 1}\sum_{l=1}^{k}\mathcal{D}_l C_l. \quad \text{(B3)}$$

From now on, we focus on the integral C_l:

$$C_l = \int_{-\infty}^{\infty} dg\frac{g_- e^{-\mu g_-^2}}{g_-^2-\alpha_{vv}^l\gamma_l}K_0(2\mu i|u|g_-). \quad \text{(B4)}$$

We reintroduce the representation $K_0(2\mu i|u|g_-)=\int_0^{\infty} ds\exp(-2\mu i|u|g_-\cosh s)$ and compute

$$C_l = \frac{1}{2\sqrt{\gamma_l\alpha_{vv}^l}}\int_0^{\infty} ds[I_+(s)-I_-(s)], \quad \text{(B5)}$$

$$I_\pm(s) = \int_{-\infty}^{\infty} dg\frac{f(g_-,s)}{g-\left(\pm\sqrt{\gamma_l\alpha_{vv}^l}+i\epsilon\right)}, \quad \text{(B6)}$$

with $f(x,s)=xe^{-\mu x^2-2\mu i|u|x\cosh s}$. The integrals $I_\pm$ are calculable by the Sokhotski-Plemelj formula:

$$I_\pm(s) = i\pi f\left(\pm\sqrt{\gamma_l\alpha_{vv}^l},s\right)+\mathrm{PV}\int_{-\infty}^{\infty}\frac{dx\, f(x,s)}{x-\left(\pm\sqrt{\gamma_l\alpha_{vv}^l}\right)}. \quad \text{(B7)}$$

The second part is the Hilbert transform [39]:

$$\frac{1}{\pi}\mathrm{PV}\int_{-\infty}^{\infty} dy\frac{ye^{-ay^2-by}}{y-x} = \frac{1}{\sqrt{a\pi}}e^{b^2/4a}+ixe^{-x^2a-xb}\mathrm{erf}\left(\frac{i}{2\sqrt{a}}(b+2ax)\right). \quad \text{(B8)}$$

Lastly, we need the identity

$$\int_x^{\infty} dt\, e^{-a^2t^2-b^2/t^2} = \frac{\sqrt{\pi}}{4a}[e^{2ab}\mathrm{erfc}(ax+b/x)+e^{-2ab}\mathrm{erfc}(ax-b/x)], \quad \text{(B9)}$$

valid for $x>0$. Combining the formulas of (B7)–(B9) results in

$$C_l = 2i\sqrt{\pi\mu}|u|e^{-\mu\alpha_{vv}^l\gamma_l}\times\int_0^{\infty} ds\int_1^{\infty} dt\cosh s\, e^{\frac{\mu\alpha_{vv}^l\gamma_l}{t^2}-\mu|u|^2t^2\cosh^2 s}. \quad \text{(B10)}$$

In the next step, we integrate over s and change the variables $t^2=\tau+1$:

$$C_l = \frac{i\pi}{2}\int_0^{\infty} d\tau\frac{1}{\tau+1}e^{-\mu|u|^2(\tau+1)-\mu\gamma_l\alpha_{vv}^l\frac{\tau}{\tau+1}}. \quad \text{(B11)}$$

We introduce a succinct contour integral representation:

$$\lim_{\gamma_1\ldots\gamma_k\to 1}\sum_{l=1}^{k}\mathcal{D}_l e^{-\mu\gamma_l\alpha_{vv}^l\frac{\tau}{\tau+1}} = \frac{1}{2\pi i}\oint_{\Gamma_s'} dq\frac{e^{-\mu q\frac{\tau}{\tau+1}}}{\prod_{i=1}^{k}\left(q-\alpha_{vv}^i\right)^{m_i}},$$

where the contour Γ_s' encircles all α_{vv}^i's counterclockwise. This formula is a part of Eq. (B3), which, after changing $p=-\mu q$, is equal to

$$J_{\vec{m}} = \frac{i\pi}{2}(-\mu)^{|\vec{m}|-1}e^{-\mu|u|^2}\times\frac{1}{2\pi i}\int_0^{\infty} d\tau\oint_{\Gamma_s} dp\frac{1}{\tau+1}\frac{e^{-\mu|u|^2\tau+\frac{p\tau}{\tau+1}}}{\prod_{i=1}^{k}\left(p+\mu\alpha_{vv}^i\right)^{m_i}}, \quad \text{(B12)}$$

with appropriately modified contour Γ_s. Lastly, we use an integral representation of the Tricomi confluent hypergeometric function:

$$\int_0^{\infty} d\tau\frac{1}{\tau+1}e^{-\mu|u|^2\tau+\frac{p\tau}{\tau+1}} = \sum_{k=0}^{\infty}U_{k+1,1}(\mu|u|^2)p^k,$$

and we combine it with Eqs. (B1) and (B12):

$$j_{\vec{m}} = \frac{\prod_{i=1}^{|\vec{m}|}(m_i-1)!}{2\pi i}\oint_{\Gamma_s} dp\sum_{k=0}^{\infty}\frac{U_{k+1,1}(\mu|u|^2)p^k}{\prod_{i=1}^{|\vec{m}|}\left(p+\mu\alpha_{vv}^i\right)^{m_i}},$$

which is exactly the formula (28).

APPENDIX C: DETAILS FOR THE CASE OF NON-NORMAL S

The ratio for the non-normal case is given by Eq. (33) with R_i terms:

$$\begin{aligned}
R_0 &= 2(Vi_{N-3,1}j_{N,1} + Zi_{N,-1}j_{N-3,2}) + 6(Vi_{N-1,0}j_{N-4,3} + Zi_{N-4,1}j_{N-1,1}) - 4Vj_{N-2,2}\delta_1^+ - 4Zi_{N-2,0}\sigma_1^+ \\
&\quad + N^2\big[j_{N-1,1}\Delta^Z_{N-3,1} + Vi_{N-3,1}j_{N,1}\big] + \mu d_1[(N-2)j_{N-1,1}i_{N-3,1} + 2i_{N-1,0}j_{N-3,2}] - \mu^2 i_{N-2,1}j_{N-2,1} \\
&\quad + N\big[2Vj_{N-2,2}\delta_1^+ - 2Zj_{N-1,1}\delta_2 + 2j_{N-3,2}\Delta^Z_{N-1,0} - 2j_{N-1,1}\Delta^Z_{N-3,1} - Zi_{N-4,1}j_{N-1,1} - 3Vi_{N-3,1}j_{N,1}\big], \\
R_1 &= -N\big[\delta_1^-\Sigma^V_{N-2,2} + \Delta^Z_{N-2,0}\sigma_1^-\big] + \mu\big[2\Delta^Z_{N-1,0}\Sigma^V_{N-3,2} + d_2 i_{N-2,0}j_{N-2,2}\big] + i_{N-1,0}(2Vj_{N-1,2} + 3j_{N-4,3}) \\
&\quad + d_1\big[2Nj_{N-2,2}\Delta^Z_{N-2,0} + i_{N-2,0}(4Vj_{N-3,3} - Nj_{N-2,2}) + i_{N-2,0}j_{N-4,3} - i_{N,-1}j_{N-2,2} + V(N-2)i_{N-2,0}j_{N-1,2} \\
&\quad - Z(N+2)i_{N-3,0}j_{N-2,2}\big] + 2Vj_{N-4,3}\delta_3 - Zi_{N,-1}\sigma_2 - 2i_{N-3,1}\Sigma^V_{N-2,2} - 2j_{N-1,1}\Delta^Z_{N-2,0} - 2Zj_{N-3,2}\Delta^Z_{N-1,-1} \\
&\quad + 2Vi_{N-1,0}\Sigma^V_{N-3,3} + j_{N-3,2}(2Zi_{N-3,0} + i_{N,-1}) - (Vi_{N-2,0} - Zi_{N,-1})j_{N-4,3}, \\
R_2 &= d_1\big[\Delta^Z_{N-1,-1}j_{N-2,2} + (i_{N-2,0} - 2i_{N,-1})\Sigma^V_{N-3,3}\big] + 2(N-2)\Delta^Z_{N-2,0}\Sigma^V_{N-2,2} + Vi_{N-2,0}\Sigma^{-V}_{N-3,3} \\
&\quad - 2(Z+V)j_{N-4,3}\Delta^Z_{N-1,-1} - \Sigma^{-Z}_{N-3,2}\Delta^Z_{N-1,-1} - i_{N-1,0}\Sigma^V_{N-3,3} + j_{N-2,2}i_{N-2,0} - \delta_3\sigma_2, \\
R_3 &= -\delta_3\Sigma^V_{N-3,3} + \Delta^Z_{N-1,-1}\big[\sigma_2 + 2d_1\Sigma^V_{N-3,3}\big], \\
R_4 &= \Delta^Z_{N-1,-1}\Sigma^V_{N-3,3},
\end{aligned}$$

where $V = |v|^2$, $Z = |z|^2$, $d_1 = \bar{z}v + z\bar{v}$, $d_2 = (\bar{z}v)^2 + (z\bar{v})^2$, and the notation reads

$$\begin{gathered}
\delta_1^\pm = i_{N-1,0} \pm i_{N-3,1}, \quad \delta_2 = i_{N-4,1} - i_{N-2,0}, \\
\delta_3 = i_{N,-1} - i_{N-2,0}, \\
\sigma_1^\pm = j_{N-3,2} \pm j_{N-1,1}, \quad \sigma_2 = j_{N-2,2} - j_{N-4,3}, \\
\Delta^z_{x,y} = i_{x,y} + zi_{x-1,y}, \quad \Sigma^z_{x,y} = j_{x,y} + zj_{x+1,y}.
\end{gathered}$$

Now we turn to the calculation of the modified bosonic block $\tilde{i}_{k,-1}$ of (38). We start from the definition (34):

$$i_{k,-1} = \frac{(-1)^k}{\mu^{k-1}}e^{-\mu|w|^2}\int_0^\infty d\rho\, e^{-\rho} I_0(2\sqrt{\mu\rho}|w|)\frac{(\rho+\mu|z|^2)^k}{(\rho+\mu k_z^+)(\rho+\mu k_z^-)}.$$

First, we express the denominator as an integral:

$$\frac{1}{(\rho+\mu k_z^+)(\rho+\mu k_z^-)} = \frac{1}{2\mu\delta k}\int_0^\infty dp\, e^{-p\rho - p\mu k_0}\sinh(p\mu\delta k),$$

with $k_z^\pm = k_0 \pm \delta k$. We consider the integral

$$\mathcal{I}(p) = \int_0^\infty d\rho\, e^{-(1+p)\rho}(\rho+\mu|z|^2)^k I_0(2\sqrt{\mu\rho}|w|) = e^{\frac{\mu|w|^2}{p+1}}\frac{(\mu|z|^2)^k k!}{p+1}\sum_{l=0}^k \frac{[\mu|z|^2(p+1)]^{-l}}{(k-l)!}L_l\left(-\frac{\mu|w|^2}{p+1}\right),$$

and we obtain the formula for $i_{k,-1}$:

$$i_{k,-1} = \frac{(-1)^k}{2\mu^k\delta k}e^{-\mu|w|^2}\int_0^\infty dp\, e^{-p\mu k_0}\sinh(p\mu\delta k)\mathcal{I}(p).$$

It gets simplified in the $w \to 0$ limit:

$$\tilde{i}_{k,-1} = \frac{(-1)^k k!}{2\mu^k\delta k}\sum_{l=0}^k \frac{(\mu|z|^2)^l}{l!}[U_{1,1+l-k}(\mu k_z^-) - U_{1,1+l-k}(\mu k_z^+)],$$

thus reproducing Eq. (38).

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