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2016 J. Phys. A: Math. Theor. 49 015201

(http://iopscience.iop.org/1751-8121/49/1/015201)

IOP Publishing Journal of Physics A: Mathematical and Theoretical
J. Phys. A: Math. Theor. **49** (2016) 015201 (18pp) doi:10.1088/1751-8113/49/1/015201

Diffusion method in random matrix theory

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Received 23 June 2015, revised 19 October 2015
Accepted for publication 28 October 2015
Published 24 November 2015

Abstract

We introduce a calculational tool useful in computing ratios and products of characteristic polynomials averaged over Gaussian measures with an external source. The method is based on Dyson's Brownian motion and Grassmann/complex integration formulas for determinants. The resulting formulas are exact for finite matrix size N and form integral representations convenient for large N asymptotics. Quantities obtained by the method are interpreted as averages over standard matrix models. We provide several explicit and novel calculations with special emphasis on the $\beta = 2$ Girko-Ginibre ensembles.

Keywords: random matrix theory, characteristic polynomials, diffusion equation

1. Introduction

One of the strengths of random matrix theory lies in the abundance of calculational tools, with the method of orthogonal polynomials [1], supersymmetric techniques [2, 3], and free probability [4, 5] among many others. This paper attempts to enlarge this family with a technique we call the diffusion method. It serves as a framework for dealing with the powers and ratios of characteristic polynomials averaged over Gaussian measures with an external source. It began as a byproduct of considerations in quantum chromodynamics (hereafter QCD) made several years ago [6] and was thereafter successfully applied to Hermitian, Wishart and chiral models [7–9]. The method uses a Dyson-like picture of dynamical matrices and Grassmann/complex integral representation of determinants.

Studying characteristic polynomials in the random matrix theory (hereafter RMT) community is now a prolific topic with many branches, but its root can be traced back to a remarkable formula relating a characteristic polynomial averaged over a $\beta = 2$ Gaussian ensemble or Gaussian unitary ensemble (hereafter GUE) to a corresponding orthogonal polynomial:

$$C_N^{-1} \int dH e^{-\mathrm{Tr}\, H^2} \det(z - H) = \pi_N(z),$$

where H is an $N \times N$ Hermitian matrix, $C_N = \int dH e^{-\mathrm{Tr}\, H^2}$ is a normalization constant, and $\pi_N(z)$ is a monic Hermite polynomial satisfying the orthogonality relation $\int dx e^{-x^2} \pi_n(x) \pi_m(x) = \delta_{nm} \sqrt{\pi} k! / 2^k$. Such objects for $\beta = 1, 2$ Gaussian ensembles were considered in [10, 27] and in many areas of application such as zeroes of Riemann ζ function [12], eigenvalue statistics in quantum chaotic systems [13], and matrix models of QCD [14]. Moreover, products and ratios of characteristic polynomials reveal rich mathematical structures in both Gaussian orthogonal, and unitary and symplectic ensembles (corresponding to $\beta = 1, 2, 4$ and hereafter abbreviated by GOE, GUE, GSE) [15–17] and $\beta = 1, 2$ Girko-Ginibre ensembles (abbreviated as GGE_β) [18–21, 37].

At the core of the method lies a seminal work of Dyson [22] who observed that a static matrix model of GOE, GUE, or GSE can also be interpreted as a dynamical system. He showed that the joint probability density function for N eigenvalues behaves exactly like a statistical system of N 'particles' interacting via the logarithmic potential. The system thus undergoes a Dysonian Brownian motion defined by the Langevin equation of the form:

$$d\lambda_i = \sum_{j(\neq i)} \frac{1}{\lambda_i - \lambda_j} dt + W_i dt,$$

where $W_i(t)$ is a delta-correlated, zero-mean Gaussian stochastic process $\left\langle W_i(t) W_j(t') \right\rangle = \delta_{ij} \delta(t - t')$. This Brownian motion of eigenvalues λ_i is induced by a Gaussian diffusion applied independently to all matrix entries. In this paper, this type of dynamics is called *entrywise* diffusion. Studying RMT from the Brownian motion's point of view has attracted attention of physicists [23, 24] and mathematicians [25, 26] alike.

The method is as discussed follows—we introduce *entrywise* dynamics to a matrix M and consider an averaged quantity of choice (i.e., the product or ratio of characteristic determinants but possibly others may apply) dependent on both M and parameter Λ_0. Then, upon proper deformation $\Lambda_0 \to \Lambda$, we find a dual- diffusion equation of this quantity in the Λ-space, which is in turn solved easily. In the end, we perform the undeformed limit.

A phenomenon where the dynamics on M induces dual dynamics in some other parameters is generally known as duality and can be found when statistical quantities are characterized by two kinds of variables—random M over which the average is taken and fixed parameters Λ_0 (i.e., the argument z in the characteristic polynomial $\det(z - M)$). It is also a general feature of RMT models that these two groups are dual or interchangeable (i.e., averages over M with fixed Λ_0 can be related to averages over Λ_0 with fixed M [30, 38]).

The main advantage of this approach lies in the fact that the dual equation has considerably lower dimension and is solved readily by heat kernel techniques. It is also readily generalized to multi-matrix models (see section 3.4) and has a built-in external source matrix models.

This paper is organized as follows. In section 2 we discuss the method's framework—constructing the dual diffusion equation and the deformation parameters. We comment on the properties and limitations of the method and establish a relation to standard random matrix models. In section 3 we calculate five examples with special attention given to $\mathrm{GGE}_{\beta=2}$. We show how to arrive at the known formula for the ratio of characteristic polynomials averaged over GUE with an external source and derive a novel duality-type equation for averaged products of characteristic polynomials in the $\mathrm{GGE}_{\beta=2}$. Furthermore, we compute a new integral representation of the averaged characteristic polynomial in $\mathrm{GGE}_{\beta=2}$ with variance structure, compute the same object for a multiplication of two $\mathrm{GGE}_{\beta=2}$ matrices, and study a

$\mathrm{GGE}_{\beta=1}/\mathrm{GGE}_{\beta=2}$ crossover model. The examples obtained provided in the last section are mostly novel results.

2. Diffusion method

We introduce an *entrywise* diffusive dynamics to M—an $N \times N$ matrix of interest. Well-suited formalism for our purpose is the multidimensional heat equation:

$$\partial_\tau P(M, \tau) = \frac{1}{N}\Delta_M P(M, \tau), \tag{1}$$

where $P(M, \tau)$ is the joint probability density function, Δ_M denotes the Laplace operator over independent degrees of freedom of M, and the constant $1/N$ is a convention. For concreteness, we list Laplace operators realizing the canonical triad of GOE, GUE, and GSE:

$$\Delta_{GOE}^{\beta=1} = \sum_{i=1}^{N} \partial_{x_{ii}}^2 + \frac{1}{2}\sum_{\substack{i,j=1\\ i<j}}^{N} \partial_{x_{ij}}^2, \qquad M_{kl} = x_{kl},\, x_{kl} = x_{lk}, \tag{2}$$

$$\Delta_{GUE}^{\beta=2} = \frac{1}{2}\sum_{i=1}^{N}\partial_{x_{ii}}^2 + \frac{1}{4}\sum_{\substack{i,j=1\\ i<j}}^{N}\left(\partial_{x_{ij}}^2 + \partial_{y_{ij}}^2\right), \qquad M_{kl} = x_{kl} + iy_{kl},\ \begin{cases} x_{kl} = x_{lk} \\ y_{kl} = -y_{lk} \end{cases}, \tag{3}$$

$$\Delta_{GSE}^{\beta=4} = \frac{1}{4}\sum_{i=1}^{N}\partial_{x_{ii}}^2 + \frac{1}{8}\sum_{\substack{i,j=1\\ i<j}}^{N}\left(\partial_{x_{ij}}^2 + \partial_{y_{ij}}^2 + \partial_{u_{ij}}^2 + \partial_{v_{ij}}^2\right),$$
$$M_{kl} = \begin{pmatrix} x_{kl} + iy_{kl} & u_{kl} + iv_{kl} \\ -u_{kl} + iv_{kl} & x_{kl} - iy_{kl} \end{pmatrix},\ \begin{cases} x_{kl} = x_{lk} \\ y_{kl} = -y_{lk} \\ u_{kl} = -u_{lk} \\ v_{kl} = -v_{lk} \end{cases}, \tag{4}$$

where the symmetries arise from the Hermiticity condition $M = M^\dagger$. The family of $\mathrm{GGE}_{\beta=1,2,4}$ read:

$$\Delta_{GGE}^{\beta=1} = \frac{1}{4}\sum_{i,j=1}^{N}\partial_{x_{ij}}^2, \qquad M_{kl} = x_{kl}, \tag{5}$$

$$\Delta_{GGE}^{\beta=2} = \frac{1}{4}\sum_{i,j=1}^{N}\left(\partial_{x_{ij}}^2 + \partial_{y_{ij}}^2\right), \qquad M_{kl} = x_{kl} + iy_{kl}, \tag{6}$$

$$\Delta_{GGE}^{\beta=4} = \frac{1}{4}\sum_{i,j=1}^{N}\left(\partial_{x_{ij}}^2 + \partial_{y_{ij}}^2 + \partial_{u_{ij}}^2 + \partial_{v_{ij}}^2\right), \qquad M_{kl} = \begin{pmatrix} x_{kl} + iy_{kl} & u_{kl} + iv_{kl} \\ -u_{kl} + iv_{kl} & x_{kl} - iy_{kl} \end{pmatrix}. \tag{7}$$

All the instances of Δ_M are Gaussian, since in this work we do not consider non-Gaussian ensembles.

The objects of interest are the ratios and products of characteristic polynomials denoted as $D(Z, M)$. For example, we study in section 3.1 an object $D(Z, M) = \frac{\det(z - M)}{\det(w - M)}$ with $Z = \{z, w\}$. We are interested in formulas for the average

$$\overline{D_\tau}(Z) := \left\langle D\left(Z, M_\tau\right)\right\rangle_{M_\tau} = C^{-1}\int dMP(M, \tau)D(Z, M), \tag{8}$$

where $\langle\rangle_{M_\tau}$ is a normalized averaging over the dynamical matrices M_τ and C is the normalization constant. The second equality is a consequence of the definition of the normalized joint probability density function $P(M, \tau) = \langle\delta(M_\tau - M)\rangle_{M_\tau}$.

To proceed, we extend $D(Z, M) \to D(Z, M; \Lambda)$ by introducing parameter-like variables Λ such that $\lim_{\Lambda\to\Lambda_0} D(Z, M; \Lambda) = D(Z, M)$ with $\Lambda_0 = 0$ in most cases. Even though the parameters Z are kept distinct from Λ, this division is purely conventional. At this point the deformation is defined in an abstract way but an algorithm for constructing Λ's is discussed in section 2.1. However, the purpose of this extension is clear—we search for a dual diffusive equation for the averaged deformed quantity $\overline{D_\tau}(Z; \Lambda) = \langle D(Z, M_\tau; \Lambda)\rangle_{M_\tau}$ in the Λ-parameter space.

In order to find it, we consider a time derivative of $\overline{D_\tau}$:

$$\partial_\tau\overline{D_\tau} = \frac{1}{N}\int dM\Delta_M P(M, \tau)D(Z, M; \Lambda) = \frac{1}{N}\int dMP(M, \tau)\Delta_M D(Z, M; \Lambda), \tag{9}$$

where we use equation (1) and integrated by parts to move the differential operator to D. Note that for Gaussian Δ_M (i.e., containing only second derivatives) integration by parts is tractable and does not produce any boundary terms for well-behaving functions P and D. The remaining task is to find $\mathbf{\Delta}_\Lambda$ such that the condition

$$\Delta_M D(Z, M; \Lambda) = \mathbf{\Delta}_\Lambda D(Z, M; \Lambda) \tag{10}$$

is satisfied. We then write the dual diffusive equation as

$$\partial_\tau\overline{D_\tau}(Z; \Lambda) = \frac{1}{N}\mathbf{\Delta}_\Lambda\overline{D_\tau}(Z; \Lambda). \tag{11}$$

As can be seen from condition (10), the Gaussian Laplace operators acting on the M manifold are transformed into Gaussian Laplace operators on the Λ space but, at the same time, we observe a decrease in the number of variables. This fact enables us to solve an initial value problem with a heat kernel K_τ:

$$\overline{D_\tau}(Z; \Lambda) = K_\tau\left(\Lambda, \Lambda'\right) \circ \overline{D_{\tau=0}}\left(Z; \Lambda'\right), \tag{12}$$

where '$\circ$' denotes a convolution operator and K_τ is defined by $\left(\partial_\tau - \frac{1}{N}\mathbf{\Delta}_\Lambda\right)K_\tau = 0$, $\lim_{\tau\to0} K_\tau(\Lambda, \Lambda') = \delta(\Lambda - \Lambda')$. As a last step, the undeformed average is

$$\overline{D_\tau}(Z) = \lim_{\Lambda\to\Lambda_0} K_\tau\left(\Lambda, \Lambda'\right) \circ \overline{D_{\tau=0}}\left(Z; \Lambda'\right). \tag{13}$$

Concrete forms of K_τ are known once we specify the problem at hand.

2.1. Constructing Λ deformations

Until now we have described how to arrive at the diffusion equation (11) in the Λ-space. Now we turn to a procedure for finding a particular deformation Λ.

We start by opening the undeformed object $D(Z, M)$ with the use of the Grassmann/complex representation of determinants:

$$\det M \sim \int d\eta d\bar\eta \exp\left(\sum_{i,j=1}^N \bar\eta_i M_{ij}\eta_j\right), \qquad \frac{1}{\det M} \sim \int d\alpha \exp\left(\sum_{i,j=1}^N \bar\alpha_i M_{ij}\alpha_j\right), \tag{14}$$

where the proportionality constants are not essential in what follows. The variables η_i and α_i denote, respectively, Grassmann and complex sets of variables. Now suppose the undeformed object D consists of k characteristic determinants and l inverse characteristic determinants; it is thus expressed as

$$D(Z, M) \sim \int d[\eta, \alpha] e^{T_G(Z,M;\eta,\alpha)}, \tag{15}$$

where T_G consists of k Grassmann and l complex binomials for every determinant and inverse determinant according to (14). A succinct notation for the measure reads $d[\eta, \alpha] = d\eta^{(1)}d\bar{\eta}^{(1)} \dots d\eta^{(k)}d\bar{\eta}^{(k)}d\alpha^{(1)} \dots d\alpha^{(l)}$.

With the help of (15), the action of the Laplacian Δ_M on $D(Z, M)$ is straightforward—it produces a certain polynomial U in both Grassmann and complex variables:

$$\Delta_M D(Z, M) \sim \int d[\eta, \alpha] U(\eta, \alpha) e^{T_G(Z,M;\eta,\alpha)}. \tag{16}$$

We assume that possible deformations Λ should not mix with the matrix M in the exponent T_G. Such an assumption is not restrictive since the structure of U already hints at particular types of deformations. However, now we observe that the action of Δ_M on a deformed $D(Z, M; \Lambda)$ should produce the same polynomial U albeit with a different exponent $T_G' = T_G + \delta T_G$:

$$\Delta_M D(Z, M; \Lambda) \sim \int d[\eta, \alpha] U(\eta, \alpha) e^{T_G'(Z,M;\Lambda;\eta,\alpha)}, \tag{17}$$

where δT_G is the unknown deformation part. To proceed, we now closely examine the structure of the polynomial U, which consists of terms with a general fourth-order structure:

$$a[v, w]_{nm} a\left[v', w'\right]_{n'm'}, \quad b[v, w]_{nm} b\left[v', w'\right]_{n'm'}, \quad c[v, w]_{nm} c\left[v', w'\right]_{n'm'} \tag{18}$$

where $a[v, w]_{nm} = \sum_{i=1}^N \bar{v}_i^{(n)}\bar{w}_i^{(m)}$, $b[v, w]_{nm} = \sum_{i=1}^N v_i^{(n)}w_i^{(m)}$, and $c[v, w]_{nm} = \sum_{i=1}^N \bar{v}_i^{(n)}w_i^{(m)}$ with variables v, w denoting either Grassmann η or complex α variables. The upper indices range over $n, m = 1 \dots (k, l)$ and the choice of v, w is only restricted so that the whole term has even Grassmann variables (i.e., is of bosonic nature). The unknown deformation is therefore given by

$$\delta T_G = \sum_{v,w=\{\eta,\alpha\}} \sum_{m,n} \left(\left(\lambda^a\right)_{mn} a[v, w]_{mn} + \left(\lambda^b\right)_{mn} b[v, w]_{mn} + \left(\lambda^c\right)_{mn} c[v, w]_{mn} \right), \tag{19}$$

where the λ parameters need to be chosen such that the whole term is of bosonic nature (see the example in section 3.1 where the deformation parameters are fermionic in nature). This general form of δT_G is evident by observing that second-order differentiation wrt. λ's produce the fourth-order terms of type (18). Therefore, along with specifying δT_G, by such considerations we also construct the operator $\boldsymbol{\Delta}_\Lambda$. The choice of non-zero parameters λ in turn forms a deformation D that satisfies the condition (10) and so the averaged quantity satisfies a dual diffusion equation (11).

By considering many examples, we have found that only the terms of c-type are present in the $\beta = 2$ cases, whereas in the $\beta = 1, 4$ a, b-terms also form the polynomial U. To make this distinction explicit, we recall the definition of an undeformed D, which, after expanding the determinants, is also expressible as a large $(Nk + Nl) \times (Nk + Nl)$ superdeterminant of a diagonal supermatrix:

$$D \sim \operatorname{sdet}\left[\operatorname{diag}\left(w_1 - M, w_2 - M, \dots, w_l - M; z_1 - M, z_2 - M, \dots z_k - M\right)\right],$$

with the $Nk \times Nk$ fermionic-fermionic and $Nl \times Nl$ bosonic-bosonic blocks. In this interpretation, off-diagonal terms are expressible by c-type terms but a, b-type terms do not fit into this structure. This argument shows why we did not address the $\beta \neq 2$ cases—they are feasible but the results are harder to calculate since we lose determinantal structures on the dual side.

2.2. Relating diffusive dynamics to random matrix models

So far we have discussed a general framework in the diffusive language. Here we comment on how to connect this approach to static random matrix models usually considered in the RMT context. An *entrywise* diffusion (1) is, as a multidimensional heat equation, reinforced with an initial condition of a delta function type $(P^i)_\beta(M, \tau \to 0) = \delta(M - M_0)$. We thus solve it for the joint probability density function P with the Laplace operators given by (2)–(4):

$$\left(P^I\right)_\beta(M, \tau) = \left(C^I\right)_\beta^{-1} \exp\left(-\frac{N\beta}{4\tau} \operatorname{Tr}\left(M - M_0\right)^2\right), \tag{20}$$

where GOE ($\beta = 1$), GUE ($\beta = 2$), and GSE ($\beta = 4$) arise, respectively, and $(C^I)_\beta$ is the normalization constant. Likewise, plugging in the operators (5)–(7) of $\mathrm{GGE}_{\beta=1,2,4}$ forms the following joint probability density functions:

$$\left(P^{II}\right)_\beta(M, \tau) = \left(C^{II}\right)_\beta^{-1} \exp\left(-\frac{N}{\tau} \operatorname{Tr}\left(M - M_0\right)^\dagger\left(M - M_0\right)\right), \tag{21}$$

where $X^\dagger \to X^T$ for $\beta = 1$ and $X^\dagger \to ZX^TZ^T$ for $\beta = 4$, where $Z = \oplus_{i=1}^N \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $(C^{II})_\beta$ is the proportionality constant. Such random matrix models dependent on a fixed matrix M_0 are called models with an external source or shifted mean models [27]. Equivalently, the matrix at time τ is equal to

$$M_\tau = M_0 + \sqrt{\tau}\mathcal{M},$$

where $\mathcal{M}$ is a matrix chosen randomly from the respective joint probability density function $(P^i)_\beta(\mathcal{M}, \tau = 1; \mathcal{M}_0 = 0)$ at vanishing M_0 and fixed time $\tau = 1$. We thus conclude that averaging over dynamical matrices M_τ is equivalent to matrix models of variance proportional to τ with an external source M_0 applied.

2.3. General properties and resume

The method is applicable to general Gaussian *entrywise* diffusion (1) with examples given in equations (2)–(7). In addition to these canonical instances, in the example of section 3.3 we enlarge this family to include Gaussian diffusion with variance structure. A dual diffusion equation (11) in the parameter space has in general lower dimensionality when compared to the matrix size and is solved readily by heat kernel techniques. Because of the underlying diffusion process, the method has a built-in initial matrix M_0 translated into an external source considered in the standard random matrix models. The final formulas also can be viewed as integral representations convenient for large N analysis.

A general way to proceed follows these subsequent steps:

1. Introduce an *entrywise* diffusion of choice (1) (see the examples of (2)–(7))
2. Define object of interest D (i.e., product and ratios of determinants) and form a Λ parameter extension D according to section 2.1
3. Infer a diffusion equation in the Λ space for the averaged quantity $\overline{D}$ with the condition (10)
4. Solve the equation (11) using the heat kernel technique and set Λ parameters to its undeformed values Λ_0 to recover the object of interest

3. Examples

This section is devoted to several examples and serves as a tour-de-force showing the framework at work to calculate new results and compare to known ones. The majority of them deal with $\beta = 2$ Girko-Ginibre ensembles.

Example 1 is devoted to probably the most thoroughly studied Gaussian unitary ensemble. We show the applicability of our method to the averaged ratio of determinants, obtain an integral representation for any external source M_0, and show how it reduces to known results [16] for $M_0 \to 0$.

Example 2 elucidates on a certain duality-type formula for $\beta = 2$ Girko-Ginibre Ensemble, a result that continues the successful program of dualities obtained in both GUE [28, 30] and GGE [18].

Example 3 is a calculation of a $\beta = 2$ Girko-Ginibre ensemble with variance structure, a model considered in [31] and inspired by the doubly-correlated Wishart ensemble [32, 33]. We compute an integral representation and compare it to known results in the vanishing external source limit.

Example 4 serves as a proof-of-concept in applying the method to the multiplication of independent matrices drawn from the $\beta = 2$ Girko-Ginibre ensemble, which has attracted a lot of attention recently [34, 39, 40]. We calculate an integral representation for the averaged characteristic polynomial.

Our last example is a toy model used to study the crossover between $\beta = 1$ and a $\beta = 2$ Girko-Ginibre ensemble inspired by elliptic ensemble [41] modeling in a similar way as the GUE-GGE$_{\beta=2}$ transition. We arrive at the large N formula of the real-axis bump developed as we vary the crossover parameter.

3.1. Ratio of determinants for $\beta = 2$ Gaussian ensemble

In this example we calculate explicit formulas for the averaged ratio of determinants by the diffusion method for the GUE. For the Laplace operator of (3), an *entrywise* diffusion equation reads:

$$\partial_\tau P(M, \tau) = \frac{1}{2N}\left(\sum_{k=1}^{N}\partial^2_{x_{kk}} + \frac{1}{2}\sum_{\substack{i,j=1\\ i>j}}^{N}\left(\partial^2_{x_{ij}} + \partial^2_{y_{ij}}\right)\right)P(M, \tau),$$

where $M_{kl} = x_{kl} + iy_{kl}$ and $x_{kl} = x_{lk}$, $y_{kl} = -y_{lk}$. We consider the ratio of characteristic polynomials:

$$D(z, w, M) = \frac{\det(z - M)}{\det(w - M)}, \tag{22}$$

which is re-expressed using (14) as

$$D(z, w, M) \sim \int d[\eta, \alpha] e^{T_G},$$
$$T_G = \sum_{k=1}^{N} \left(x_{kk}(\bar{\eta}_k \eta_k + \bar{\alpha}_k \alpha_k) - z\bar{\eta}_k \eta_k - w\bar{\alpha}_k \alpha_k \right) + \sum_{\substack{k,l=1\\k<l}}^{N} x_{kl}(\bar{\eta}_k \eta_l - \eta_k \bar{\eta}_l + \bar{\alpha}_k \alpha_l + \alpha_k \bar{\alpha}_l) + i \sum_{\substack{k,l=1\\k<l}}^{N} y_{kl}(\bar{\eta}_k \eta_l + \eta_k \bar{\eta}_l + \bar{\alpha}_k \alpha_l - \alpha_k \bar{\alpha}_l).$$

We construct the deformation Λ following the steps given in section 2.1. First, the quantity $\Delta_M D(z, w, M)$ is calculated and the polynomial U of (16) is identified as

$$U = \sum_{i=1}^{N} \bar{\eta}_i \eta_i \bar{\alpha}_i \alpha_i + \frac{1}{2} \sum_{i=1}^{N} \bar{\alpha}_i{}^2 \alpha_i^2 + \sum_{\substack{i,j=1\\i<j}}^{N} \left(\alpha_i \bar{\alpha}_j - \eta_i \bar{\eta}_j \right) \left(\bar{\eta}_i \eta_j + \bar{\alpha}_i \alpha_j \right)$$
$$= - \sum_{\substack{i,j=1\\i<j}}^{N} \bar{\eta}_i \eta_i \bar{\eta}_j \eta_j + \frac{1}{2} \sum_{i=1}^{N} \bar{\alpha}_i{}^2 \alpha_i^2 + \sum_{\substack{i,j=1\\i<j}}^{N} \bar{\alpha}_i \alpha_i \bar{\alpha}_j \alpha_j + \sum_{\substack{i,j=1\\i<j}}^{N} \bar{\alpha}_i \eta_i \bar{\eta}_j \alpha_j + \sum_{\substack{i,j=1\\i<j}}^{N} \bar{\eta}_i \alpha_i \bar{\alpha}_j \eta_j + \sum_{i=1}^{N} \bar{\eta}_i \eta_i \bar{\alpha}_i \alpha_i.$$

The goal is to find a deformation parameter Λ and the corresponding Laplace operator reproducing this polynomial. As a first step, we calculate two derivatives wrt. parameters z and w:

$$\partial_{zz} D \sim \int d[\eta, \alpha] \left(2 \sum_{\substack{i,j=1\\i<j}}^{N} \bar{\eta}_i \eta_i \bar{\eta}_j \eta_j \right) e^{T_G},$$
$$\partial_{ww} D \sim \int d[\eta, \alpha] \left(\sum_{i=1}^{N} \bar{\alpha}_i{}^2 \alpha_i^2 + 2 \sum_{\substack{i,j=1\\i<j}}^{N} \bar{\alpha}_i \alpha_i \bar{\alpha}_j \alpha_j \right) e^{T_G}, \tag{23}$$

which already forms first three terms of U. To obtain the remaining ones we identify two a-type quantities $a[\eta, \alpha] = \sum_{i=1}^{N} \bar{\eta}_i \alpha_i$, $a'[\alpha, \eta] = \sum_{i=1}^{N} \bar{\alpha}_i \eta_i$ and thus establish two deformation parameters p and q forming δT_G:

$$\delta T_G = - \sum_{i=1}^{N} (\bar{\eta}_i p \alpha_i + \bar{\alpha}_i q \eta_i),$$

where the structure of (19) is evident and the chosen signs are a convention. The undeformed values of p, q are 0. Both are Grassmann numbers so that the δT_G is bosonic in nature. The deformed ratio $D(z, w, M; \Lambda)$ is

$$D(z, w, M; q, p) \sim \int d[\eta, \alpha] \exp\left[-\begin{pmatrix}\bar{\alpha} & \bar{\eta}\end{pmatrix}\begin{pmatrix} w - M & q \\ p & z - M \end{pmatrix}\begin{pmatrix}\alpha \\ \eta\end{pmatrix}\right]$$
$$= \int d[\eta, \alpha] e^{T_G'} = \operatorname{sdet}\begin{pmatrix} w - M & q \\ p & z - M \end{pmatrix},$$

where $T_G' = T_G + \delta T_G$ and the terms proportional to p and q form off-diagonal parts of the supermatrix. We calculate that

$$\partial_p \partial_q D \sim \int d[\eta, \alpha] \left(-\sum_{\substack{i,j=1 \\ i<j}}^{N} \bar{\alpha}_i \eta_i \bar{\eta}_j \alpha_j - \sum_{\substack{i,j=1 \\ i<j}}^{N} \bar{\eta}_i \alpha_i \bar{\alpha}_j \eta_j - \sum_{i=1}^{N} \bar{\eta}_i \eta_i \bar{\alpha}_i \alpha_i \right) e^{T_G'}$$

reproduces the remaining part of U and thus forms, together with (23), the Laplace operator in the parameter space:

$$\boldsymbol{\Delta}_\Lambda = \frac{1}{2}\left(\partial_{ww} - \partial_{zz} - 2\partial_p \partial_q\right).$$

The dual diffusion-like equation (11) is equal:

$$\partial_\tau \overline{D_\tau}(z, w; p, q) = \frac{1}{2N}\left(\partial_{ww} - \partial_{zz} - 2\partial_p \partial_q\right)\overline{D_\tau}(z, w; p, q). \tag{24}$$

We comment on two features of (24)—in the z direction it has a negative diffusivity constant and the diffusion also occurs in the p, q Grassmann 'directions.' In the RMT context the negative diffusive constant is interpreted as a source of a universal oscillatory behavior [9]. To deal with it on a technical level we can either Wick rotate the $z \to iz$ variable or consider instead a modified object $\frac{\det(iz - M)}{\det(w - M)}$, and we choose the former approach since it is more intuitive. In considering Grassmann 'diffusion' we make use of the well-known property of superdeterminants:

$$\operatorname{sdet}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{\det\left(d - ca^{-1}b\right)}{\det(a)}$$

and utilize the 'flatness' property $q^2 = 0, p^2 = 0$ to expand $D(z, w, H; p, q) = D^{(1)} + pD^{(2)} + qD^{(3)} + qpD^{(4)}$ in the Grassmann parameters. We rewrite (24) as an equivalent system of four equations for each $D^{(i)}$:

$$\partial_\tau \overline{D_\tau}^{(1)} = \frac{1}{2N}\left(\partial_{ww} - \partial_{zz}\right)\overline{D_\tau}^{(1)} - \frac{1}{N}\overline{D_\tau}^{(4)}, \tag{25}$$

$$\partial_\tau \overline{D_\tau}^{(2)} = \frac{1}{2N}\left(\partial_{ww} - \partial_{zz}\right)\overline{D_\tau}^{(2)}, \tag{26}$$

$$\partial_\tau \overline{D_\tau}^{(3)} = \frac{1}{2N}\left(\partial_{ww} - \partial_{zz}\right)\overline{D_\tau}^{(3)}, \tag{27}$$

$$\partial_\tau \overline{D_\tau}^{(4)} = \frac{1}{2N}\left(\partial_{ww} - \partial_{zz}\right)\overline{D_\tau}^{(4)}. \tag{28}$$

To find the solution of (24) we observe that only equations (25) and (28) contain relevant components $i = 1, 4$ since ultimately we are interested in the undeformed limit $\lim_{p,q\to 0} D = D^{(1)}$. To solve them we form a heat kernel of the Laplace operator $\frac{1}{2N}(\partial_{ww} - \partial_{zz})$:

$$K_\tau(z, w; y, v) = \frac{N}{2\pi\tau}\exp\left(-\frac{N}{2\tau}(v-w)^2 - \frac{N}{2\tau}(y-iz)^2\right), \tag{29}$$

where the z direction is Wick rotated, and we thus form the z dependent part of the solution to (25) and (28) by analytic continuation. The solution to (28) is

$$\overline{D_\tau}^{(4)}(z, w) = \int dydvK_\tau(z, w; y, v)D_0^{(4)}\left(-iy, v; M_0\right) =: \left(K_\tau \circ D_0^{(4)}\right)(z, w),$$

with M_0 denoting the initial matrix. With this notation, the solution to the inhomogeneous heat equation (25) is

$$\overline{D_\tau}^{(1)}(z, w) = \left(K_\tau \circ \left(D_0^{(1)} - \frac{\tau}{N}D_0^{(4)}\right)\right)(z, w) = \left(K_\tau \circ D_0^{(1)}\right)(z, w) - \frac{\tau}{N}\overline{D_\tau}^{(4)}(z, w). \tag{30}$$

To write it explicitly, we expand the initial condition:

$$D = \frac{\det\left(z - M_0\right)}{\det\left(w - M_0\right)}\left(1 + qp\,\mathrm{Tr}\,\frac{1}{\left(z - M_0\right)\left(w - M_0\right)}\right),$$

$$D_0^{(1)}\left(y, v; M_0\right) = \frac{\det\left(y - M_0\right)}{\det\left(v - M_0\right)},$$

$$D_0^{(4)}\left(y, v; M_0\right) = \frac{\det\left(y - M_0\right)}{\det\left(v - M_0\right)}\,\mathrm{Tr}\,\frac{1}{\left(y - M_0\right)\left(v - M_0\right)}.$$

Due to the unitary invariance, the most general initial matrix is diagonal $M_0 = \mathrm{diag}(h_1, \ldots, h_N)$, where some values of h_i can coincide. We also form an N-dimensional indexing vector $\vec{h} = (h_1, \ldots, h_N)$ and introduce two functions:

$$\pi_{\vec{h}}(z) = \sqrt{\frac{N}{2\pi\tau}}\int due^{-\frac{N}{2\tau}(u-iz)^2}\prod_{i=1}^{N}\left(-iu - h_i\right), \tag{31}$$

$$\theta_{\vec{h}}(w) = \sqrt{\frac{N}{2\pi\tau}}\int dqe^{-\frac{N}{2\tau}(q-w)^2}\prod_{i=1}^{N}\frac{1}{\left(q - h_i\right)}. \tag{32}$$

After setting $p, q \to 0$, the averaged ratio of characteristic polynomials (30) is equal to

$$\overline{D_\tau}(z, w) = \pi_{\vec{h}}(z)\theta_{\vec{h}}(w) - \frac{\tau}{N}\sum_{i=1}^{N}\pi_{\vec{h}_-(i)}(z)\theta_{\vec{h}_+(i)}(w), \tag{33}$$

where we introduced an extended $N+1$ dimensional vector $\vec{h}_+(i) = (h_1, \ldots, h_{i-1}, h_i, h_i, h_{i+1}, \ldots, h_N)$ and contracted $N-1$ dimensional vector $\vec{h}_-(i) = (h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_N)$. To connect with known results, we write the average explicitly using (20) as

$$\overline{D_\tau}(z, w) = \left(C^I\right)_\beta^{-1}\int dHe^{-\frac{N}{2\tau}\mathrm{Tr}\left(M-M_0\right)^2\right)}\frac{\det(z - M)}{\det(w - M)}. \tag{34}$$

This quantity is present as a building block of biorthogonal structures [35, 36], where θ and π are the multiple orthogonal polynomials of type I and II, respectively.

To recover known formulas for the GUE case, we set $h_i = 0$ for all $i = 1 \ldots N$ so that $\vec{h}$, $\vec{h}_+(i)$ and $\vec{h}_-(i)$ become a N, $(N+1)$ and $N-1$ dimensional null vector, respectively. It is now more natural to introduce simplified notation: $\vec{h} \to N$, $\vec{h}_-(i) \to N-1$ and $\vec{h}_+(i) \to N+1$. Now the type I orthogonal polynomial associated with a $k \times k$ matrix is given by $\theta_k(w) = \gamma_{k-1} f_{k-1}(w)$, where $\gamma_k = \frac{1}{k!}\left(\frac{N}{\tau}\right)^k \sqrt{\frac{N}{2\pi\tau}}$ and $f_k(z) = \int \frac{e^{-\frac{N}{2\tau}s^2}}{z-s}\pi_k(s)$ is the Cauchy transform. Along with $\gamma_N/\gamma_{N-1} = \frac{1}{\tau}$, we rewrite (33) as:

$$\overline{D_\tau}(z, w) = \gamma_{N-1}\pi_N(z) f_{N-1}(w) - \tau\gamma_N f_N(w)\pi_{N-1}(z),$$

which is the ratio formula calculated for GUE in [16].

3.2. Duality formula for $\beta = 2$ Girko-Ginibre ensemble

Let $M_{kl} = x_{kl} + iy_{kl}$ be an $N \times N$ matrix. We introduce an *entrywise* diffusive dynamics with a Laplacian (6):

$$\partial_\tau P(M, \tau) = \frac{1}{4N}\sum_{i,j=1}^{N}\left(\partial_{x_{ij}}^2 + \partial_{y_{ij}}^2\right) P(M, \tau),$$

which also describes the $\mathrm{GGE}_{\beta=2}$. We aim at calculating an averaged product of k characteristic polynomials:

$$D^{(k)}(\mathcal{Z}, M) = \det\left[\prod_{i=1}^{k}\left(z_i - M\right)\left(\bar{z}_i - M^\dagger\right)\right]. \tag{35}$$

In this example we skip the procedure of constructing a deformed quantity $D^{(k)}$, which was described in section 2.1 and presented in the example of section 3.1. Deformation is a $2kN$ block matrix of the form

$$D^{(k)}(\mathcal{Z}, M; A) = \det\begin{pmatrix} \mathcal{Z}\otimes 1_N - 1_k \otimes M & -A^\dagger \otimes 1_N \\ A \otimes 1_N & \mathcal{Z}^\dagger \otimes 1_N - 1_k \otimes M^\dagger \end{pmatrix}, \tag{36}$$

where $\mathcal{Z} = \mathrm{diag}(z_1, \dots z_k)$, 1_n is an N-dimensional unit matrix and A is a complex $k \times k$ matrix, representing the Λ-parameter space. We baptize $D^{(k)}$ the k-extended averaged characteristic polynomial (k-EACP) in agreement with [20] where the authors considered a particular case of $k = 1$. In the limit $\lim_{A\to 0} D^{(k)}$ we recover (35).

To proceed, we open the $D^{(k)}$ using Grassmann variables:

$$D^{(k)}(\mathcal{Z}, M; A) \sim \int d[\eta, \xi]\; e^{T_G'},$$

$$T_G' = \sum_{i=1}^{k}\bar{\eta}^{(i)}\cdot\eta^{(i)} z_i + \sum_{i=1}^{k}\bar{\xi}^{(i)}\cdot\xi^{(i)}\bar{z}_i - \sum_{i=1}^{k}\bar{\eta}^{(i)}\cdot M\cdot\eta^{(i)} - \sum_{i=1}^{k}\bar{\xi}^{(i)}\cdot M^\dagger\cdot\xi^{(i)} \underline{- \sum_{i,j=1}^{k}\bar{\eta}^{(i)}\cdot\xi^{(j)}A_{ij}^\dagger + \sum_{i,j=1}^{k}\bar{\xi}^{(i)}\cdot\eta^{(j)}A_{ij}},$$

with kN-dimensional Grassmann vectors $\xi_j^{(i)}$ and $\eta_j^{(i)}$ $(i = 1 \dots k, j = 1 \dots N)$. We also introduced a dot '·' denoting a sum over N-dimensional indices, a notation useful in this and forthcoming examples. The underlined part forms the deformation δT_G. We find that

$$\Delta_M D^{(k)} \sim \int d[\eta, \xi]\left(\sum_{i,j=1}^{k}\eta^{(i)}\cdot\bar{\xi}^{(j)}\bar{\eta}^{(i)}\cdot\xi^{(j)}\right) e^{T_G'},$$

which determines the parameter Laplace operator as

$$\mathbf{\Delta}_{\Lambda} = \frac{1}{4}\sum_{i,j=1}^{k}\left(\partial^2_{a_{ij}} + \partial^2_{b_{ij}}\right) =: \operatorname{Tr}\partial_{AA^\dagger},$$

where $A_{kl} = a_{kl} + ib_{kl}$. We thus arrive at the final equation for the average $\langle D^{(k)}\rangle_{M_\tau} = \overline{D_\tau}^{(k)}(\mathcal{Z}; A)$:

$$\partial_\tau \overline{D_\tau}^{(k)}(\mathcal{Z}; A) = \frac{1}{N}\operatorname{Tr}\partial_{AA^\dagger}\overline{D_\tau}^{(k)}(\mathcal{Z}; A), \tag{37}$$

where we observe a dimensional reduction in diffusive variables $N \times N \to k \times k$. Using (21) and the proportionality constant $(C^{II})_2^{-1} = \left(\frac{N}{\pi\tau}\right)^{k^2}$, the solution is

$$\overline{D_\tau}^{(k)}(\mathcal{Z}; A) = \left(\frac{N}{\pi\tau}\right)^{k^2}\int dBe^{-\frac{N}{\tau}\operatorname{Tr}(B-A)\left(B^\dagger - A^\dagger\right)}D^{(k)}\left(\mathcal{Z}, M_0; B\right),$$

where A and M_0 are the initial values of the parameters- and the randomized matrix, respectively. We turn to the product of characteristic polynomials by taking the undeformed limit $A \to 0$:

$$\overline{D_\tau}^{(k)}(\mathcal{Z}) = \left(\frac{N}{\pi\tau}\right)^{k^2}\int dBe^{-\frac{N}{\tau}\operatorname{Tr}BB^\dagger}D^{(k)}\left(\mathcal{Z}, M_0; B\right). \tag{38}$$

To arrive at the duality formula, we write the definition of an average $\overline{D_\tau}^{(k)}$ using (21):

$$\overline{D_\tau}^{(k)}(\mathcal{Z}) = \left(\frac{N}{\pi\tau}\right)^{N^2}\int dMe^{-\frac{N}{\tau}\operatorname{Tr}\left(M-M_0\right)^\dagger\left(M-M_0\right)}D^{(k)}(\mathcal{Z}, M), \tag{39}$$

but this time $(C^{II})_2^{-1} = \left(\frac{N}{\pi\tau}\right)^{N^2}$. We can thus write the duality from (38) and (39):

$$\left(\frac{N}{\pi\tau}\right)^{N^2}\int dMe^{-\frac{N}{\tau}\operatorname{Tr}MM^\dagger}D^{(k)}\left(\mathcal{Z}, M + M_0; A = 0\right) = \left(\frac{N}{\pi\tau}\right)^{k^2}\int dBe^{-\frac{N}{\tau}\operatorname{Tr}BB^\dagger}D^{(k)}\left(\mathcal{Z}, M_0; B\right), \tag{40}$$

with the definition repeated for clarity:

$$D^{(k)}(\mathcal{Z}, M; A) = \det\begin{pmatrix} \mathcal{Z}\otimes 1_N - 1_k \otimes M & -A^\dagger \otimes 1_N \\ A \otimes 1_N & \mathcal{Z}^\dagger \otimes 1_N - 1_k \otimes M^\dagger \end{pmatrix}.$$

This new result is an extension of a similar formula for $M_0 = 0$ obtained in [18]. Such dual quantities were studied extensively in $\beta = 2$ Gaussian ensembles by [28], for general β in [30], and in the context of string theory by [29] among others.

3.3. $\beta = 2$ Girko-Ginibre ensemble with variance structure

In another example we deal with a $\mathrm{GGE}_{\beta=2}$ matrix model with variance structure. With $\tilde{M}_{kl} = \tilde{x}_{kl} + i\tilde{y}_{kl}$ we define it as

$$\partial_\tau \tilde{P}\left(\tilde{M}, \tau\right) = \frac{1}{4N}\sum_{i,j=1}^{N}\Gamma_{ii}^{-2}\Omega_{jj}^{-2}\left(\partial^2_{\tilde{x}_{ij}} + \partial^2_{\tilde{y}_{ij}}\right)\tilde{P}\left(\tilde{M}, \tau\right),$$

where the variance structure is assumed to be strictly positive $\Gamma_{ii}, \Omega_{jj} > 0$. The fundamental solution is

$$\tilde{P}\left(\tilde{M}, \tau\right) = \tilde{C}^{-1} \exp\left(-\frac{N}{\tau} \operatorname{Tr} \Gamma^2 \left(\tilde{M} - \tilde{M}_0 \right) \Omega^2 \left(\tilde{M}^\dagger - \tilde{M}_0^\dagger \right) \right),$$

where $\tilde{M}_0$ denotes an initial matrix and $\tilde{C}^{-1}$ is a normalization constant. For $\tilde{M}_0 = 0$ this measure is called a doubly-correlated $\beta = 2$ Wishart ensemble with both time (Γ) and space (Ω) correlations [33]. However, here we treat it as a Girko-Ginibre model (i.e., the eigenvalues of $\tilde{M}$ are investigated instead of the eigenvalues of $\tilde{M}^\dagger \tilde{M}$). A natural object of interest is a characteristic determinant:

$$\tilde{D}\left(z, \tilde{M}\right) = \det\left(z - \tilde{M}\right) \det\left(\bar{z} - \tilde{M}^\dagger\right). \tag{41}$$

It is convenient to consider a reparametrization $M = \Gamma \tilde{M} \Omega$, where the new matrix $M_{kl} = x_{kl} + i y_{kl}$ undergoes an usual *entrywise* diffusion equation (6):

$$\partial_\tau P(M, \tau) = \frac{1}{4N} \sum_{i,j=1}^{N} \left(\partial_{x_{ij}}^2 + \partial_{y_{ij}}^2 \right) P(M, \tau).$$

and the quantity of interest $\tilde{D}$ is modified to

$$\tilde{D}\left(z, \Gamma^{-1} M \Omega^{-1}\right) = D^{(\Gamma,\Omega)}(z, M) = \det\left(z - \Gamma^{-1} M \Omega^{-1}\right) \det\left(\bar{z} - \Omega^{-1} M^\dagger \Gamma^{-1}\right), \tag{42}$$

which we open using (14):

$$D^{(\Gamma,\Omega)} \sim \int d[\eta, \xi] e^{T_G},$$
$$T_G = \bar{\eta} \cdot \eta z + \bar{\xi} \cdot \xi \bar{z} - \bar{\eta} \cdot \left(\Gamma^{-1} X \Omega^{-1}\right) \cdot \eta - \bar{\xi} \cdot \Omega^{-1} M^\dagger \Gamma^{-1} \cdot \xi.$$

According to section 2.1, we look for a deformation by calculating the action of Laplacian $\Delta_M D^{(\Gamma,\Omega)}$ to obtain the polynomial U:

$$U = \sum_{i,j=1}^{N} \Gamma_{ii}^{-2} \xi_i \bar{\eta}_i \Omega_{jj}^{-2} \bar{\xi}_j \eta_j, \tag{43}$$

which depends on the variances Ω, Γ but the formula retains the structure of equation (18). Both variance matrices modify the c-type terms slightly:

$$c[v, w] = \sum_{i=1}^{N} \bar{v}_i w_i \to \sum_{i=1}^{N} V_{ii} \bar{v}_i w_i, \qquad V_{ii} = \left\{\Gamma_{ii}, \Omega_{ii}\right\},$$

but we form the δT_G out of modified c-type terms as:

$$\delta T_G = -\bar{w} \bar{\eta} \cdot \Gamma^{-2} \cdot \xi + w \bar{\xi} \cdot \Omega^{-2} \cdot \eta,$$

with an introduced Λ-parameter w and arbitrary signs. By setting $T_G' = T_G + \delta T_G$, the deformed determinant $D^{(\Gamma,\Omega)}$ is expressible as a block matrix with off-diagonal elements encoding the deformation:

$$D^{(\Gamma,\Omega)}(z, M; w) = \det \begin{pmatrix} z - \Gamma^{-1} M \Omega^{-1} & -\Gamma^{-2} \bar{w} \\ \Omega^{-2} w & \bar{z} - \Omega^{-1} M^\dagger \Gamma^{-1} \end{pmatrix}, \tag{44}$$

with $\lim_{w \to 0} D^{(\Gamma,\Omega)} = D^{(\Gamma,\Omega)}$. In the last step, we find that the action of $\partial_{\bar{w} w}$ acting on $D^{(\Gamma,\Omega)}$ reproduces the polynomial (43) and thus forms Laplacian in the parameter space:

$$\boldsymbol{\Delta}_\Lambda = \partial_{w \bar{w}},$$

and the final equation for the averaged $\overline{D_\tau}$ is

$$\partial_\tau \overline{D_\tau}(z;\, w) = \frac{1}{N} \partial_{w\bar{w}} \overline{D_\tau}(z;\, w). \tag{45}$$

Because the resulting dual equation is two-dimensional, we readily form the solution:

$$\overline{D_\tau}(z) = \frac{N}{\pi\tau} \int d^2 u e^{-\frac{N}{\tau}|u|^2} D^{(\Gamma,\Omega)}\left(z,\, M_0;\, u\right), \tag{46}$$

$$D^{(\Gamma,\Omega)}(z,\, M;\, w) = \det \begin{pmatrix} z - \Gamma^{-1} M \Omega^{-1} & -\Gamma^{-2}\bar{w} \\ \Omega^{-2} w & \bar{z} - \Omega^{-1} M^\dagger \Gamma^{-1} \end{pmatrix},$$

where we took the undeformed limit $w \to 0$. It is an integral representation valid for general M_0 and correlations Γ, Ω. For completeness, the averaged quantity $\overline{D_\tau}(z)$ is explicitly expressed with the use of the joint probability density function (21) as

$$\overline{D_\tau}(z) = \left(C^{II}\right)_2^{-1} \int dM \exp\left(-\frac{N}{\tau} \operatorname{Tr}\left(M - M_0\right)^\dagger \left(M - M_0\right)\right) D^{(\Gamma,\Omega)}(z,\, M). \tag{47}$$

In the special $M_0 \to 0$ limit, the solution (46) reproduces the result of [31]:

$$\overline{D_\tau}(z) = \frac{2N}{\tau} \int_0^\infty d\rho \rho e^{-\frac{N}{\tau}\rho^2} \prod_{i=1}^N \left(|z|^2 + \rho^2 \Gamma_{ii}^{-2} \Omega_{ii}^{-2}\right).$$

3.4. Multiplication of two independent $\beta = 2$ Girko-Ginibre matrices

In this example we show how the method is applied to a product of two $\beta = 2$ Girko-Ginibre matrices, a case that has drawn much attention recently [34, 39, 40]. We introduce two matrices, M_1, M_2, each undergoing an independent $\mathrm{GGE}_{\beta=2}$ *entrywise* diffusion of (6):

$$\begin{aligned} \partial_\tau P\left(M_1,\, M_2,\, \tau\right) = &\frac{1}{4N} \sum_{i,j=1}^N \left(\partial_{x_{ij}}^2 + \partial_{y_{ij}}^2 \right) P\left(M_1,\, M_2,\, \tau\right) \\ &+ \frac{1}{4N} \sum_{i,j=1}^N \left(\partial_{x'_{ij}}^2 + \partial_{y'_{ij}}^2 \right) P\left(M_1,\, M_2,\, \tau\right), \end{aligned}$$

where $(M_1)_{kl} = x_{kl} + iy_{kl}$ and $(M_2)_{kl} = x'_{kl} + iy'_{kl}$. We consider a determinant of the form:

$$D\left(z,\, M_1,\, M_2\right) = \det\left(z - M_1 M_2\right) \det\left(\bar{z} - \left(M_1 M_2\right)^\dagger\right). \tag{48}$$

To proceed, we linearize it by expanding the block structure accordingly:

$$\begin{aligned} D &= \det \begin{pmatrix} z - M_1 M_2 & 0 \\ 0 & \bar{z} - \left(M_1 M_2\right)^\dagger \end{pmatrix} = \det\left(\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} - \begin{pmatrix} 0 & M_1 \\ M_2^\dagger & 0 \end{pmatrix} \begin{pmatrix} 0 & M_1^\dagger \\ M_2 & 0 \end{pmatrix} \right) \\ &= \det \begin{pmatrix} z & 0 & 0 & X_1 \\ 0 & \bar{z} & M_2^\dagger & 0 \\ 0 & M_1^\dagger & 1 & 0 \\ M_2 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

where we used formulas valid for block matrices:

$$\begin{pmatrix} M_1M_2 & 0 \\ 0 & (M_1M_2)^\dagger \end{pmatrix} = \begin{pmatrix} 0 & M_1 \\ M_2^\dagger & 0 \end{pmatrix}\begin{pmatrix} 0 & M_1^\dagger \\ M_2 & 0 \end{pmatrix},$$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det(ad - bc), \qquad \text{if } cd = dc.$$

After this preparatory transformation, we identify deformation parameters as described in section 2.1. We skip this part due to its similarity to previous examples and just write the resulting deformed characteristic polynomial:

$$D(z, M_1, M_2; u, v, w) = \det\begin{pmatrix} z & -\bar{w} & 0 & M_1 \\ v & \bar{z} & M_2^\dagger & 0 \\ 0 & M_1^\dagger & u & w \\ M_2 & 0 & -\bar{v} & \bar{u} \end{pmatrix}, \tag{49}$$

where three additional parameters u, v, w were introduced. We open this determinant:

$$\begin{aligned} D &\sim \int d\xi d\eta e^{T_G'}, \\ T_G' &= \bar{\xi}_1 \cdot \xi_1 z + \bar{\xi}_2 \cdot \xi_2 \bar{z} \\ &\quad + \underline{\bar{\eta}_1 \cdot \eta_1 u + \bar{\eta}_2 \cdot \eta_2 \bar{u} - \bar{\xi}_1 \cdot \xi_2 \bar{w} + \bar{\xi}_2 \cdot \xi_1 v + \bar{\eta}_1 \cdot \eta_2 w - \bar{\eta}_2 \cdot \eta_1 \bar{v}} \\ &\quad + \bar{\xi}_1 \cdot X_1 \cdot \eta_2 + \bar{\xi}_2 \cdot X_2^\dagger \cdot \eta_1 + \bar{\eta}_1 \cdot X_1^\dagger \cdot \xi_2 + \bar{\eta}_2 \cdot X_2 \cdot \xi_1, \end{aligned}$$

where ξ_i, η_i are Grassmann variables and the underlined part forms δT_G. The joint Laplace operator acting on D is

$$(\Delta_{M_1} + \Delta_{M_1})D \sim \int d[\eta, \xi](\bar{\xi}_1 \cdot \xi_2 \eta_2 \cdot \bar{\eta}_1 + \bar{\eta}_2 \cdot \eta_1 \xi_1 \cdot \bar{\xi}_2)e^{T_G'},$$

which also dictates the Laplace operator in the parameter space to be of the form

$$\boldsymbol{\Delta}_\Lambda = \partial_{w,\bar{w}} + \partial_{v,\bar{v}}.$$

The dual diffusion equation for the averaged determinant is

$$\partial_\tau \overline{D_\tau}(z; u, v, w) = \frac{1}{N}(\partial_{w,\bar{w}} + \partial_{v,\bar{v}})\overline{D_\tau}(z; u, v, w), \tag{50}$$

We write the solution in the undeformed limit $v, w \to 0$ and $u \to 1$:

$$\overline{D_\tau}(z) = \left(\frac{N}{\pi\tau}\right)^2 \int d^2w d^2v e^{-\frac{N}{\tau}(|w|^2+|v|^2)} D\left(z, (M_1)_0, (M_2)_0; 1, v, w\right). \tag{51}$$

As before, we investigate the vanishing source limit $(M_i)_0 \to 0$, where

$$D(z, 0, 0; u, v, w) = (1 + \bar{v}w)^N (|z|^2 + v\bar{w})^N.$$

The angles of w, v in (51) can be integrated out with the help of hypergeometric function ${}_2F_1$:

$$\overline{D_\tau}(z) = \left(\frac{2N}{\tau}\right)^2 \int_0^\infty dpdq\; qpe^{-\frac{N}{\tau}\left(q^2+p^2\right)}\left(|z|^2 + q^2p^2\right)^N {}_2F_1 \\ \times \left(\frac{1-N}{2}, -\frac{N}{2}, 1, \frac{4|z|^2p^2q^2}{\left(|z|^2+p^2q^2\right)^2}\right),$$

and we simplify it further by introducing $p^2 = t\alpha$, $q^2 = \frac{t}{\alpha}$ and integrating over α's:

$$\overline{D_\tau}(z) = \left(\frac{2N}{\tau}\right)^2 \int_0^\infty dt\; tK_0\left(\frac{2Nt}{\tau}\right)\left(|z|^2 + t^2\right)^N {}_2F_1\left(\frac{1-N}{2}, -\frac{N}{2}, 1, \frac{4|z|^2t^2}{\left(|z|^2+t^2\right)^2}\right), \tag{52}$$

where K_0 is the modified Bessel function of the second kind. The average $\overline{D_\tau}$ is explicitly given as

$$\overline{D_\tau}(z) = \left(C^{II}\right)_2^{-2} \int dM_1 dM_2 \exp\left(-\frac{N}{\tau}\mathrm{Tr}\left(M_1^\dagger M_1 + M_2^\dagger M_2\right)\right) \\ \times \det\left(z - M_1M_2\right)\det\left(\bar{z} - \left(M_1M_2\right)^\dagger\right)$$

according to (21). To the best of our knowledge, this result has not been considered previously.

3.5. Girko-Ginibre ensemble crossover model between $\beta = 1$ and $\beta = 2$

The last example is a crossover model between real and complex Girko-Ginibre ensembles. A matrix drawn from $\mathrm{GGE}_{\beta=1}$ has either real or complex conjugated pairs of eigenvalues, whereas $\mathrm{GGE}_{\beta=2}$ is not constrained by such condition—its eigenvalues spread evenly over the complex plane. To study this transition, we introduce an *entrywise* diffusion combining the Laplace operators of (5) and (6):

$$\partial_\tau P(M, \tau) = \frac{1}{4N}\sum_{i,j=1}^{N}\left(\partial_{x_{ij}}^2 + \alpha^2\partial_{y_{ij}}^2\right)P(M, \tau),$$

which forms an $N \times N$ matrix $M_{kl} = x_{kl} + iy_{kl}$. The model introduces a crossover parameter α that varies between 0 $(\beta = 1)$ and $1 (\beta = 2)$. We investigate the condensation of eigenvalues on the real line as we take the limit $\alpha \to 0$. We are interested in a standard characteristic polynomial:

$$D(z, M) = \det(z - M)\det\left(\bar{z} - M^\dagger\right). \tag{53}$$

After finding the deformation analogously to the examples of section 3.2 and 3.3, we form a deformed quantity:

$$D(z, M; w) = \begin{pmatrix} z - M & -\bar{w} \\ w & \bar{z} - M^\dagger \end{pmatrix}, \tag{54}$$

for which, using the same techniques as previously, we find a dual diffusion equation:

$$\partial_\tau \overline{D_\tau}(z; w) = \frac{1+\alpha^2}{2N}\partial_{w\bar{w}}\overline{D_\tau}(z; w). \tag{55}$$

The solution, after taking the w $\to$ 0 limit, is

$$\overline{D_\tau}(z) = \frac{2N}{\tau}\int_0^\infty dr r e^{-\frac{2N}{\tau(1+\alpha^2)}r^2} D\left(z, M_0; r\right), \tag{56}$$

which is valid for any external source M_0. For vanishing external source $M_0 \to 0$, the formula (56) agrees with the results for both $\mathrm{GGE}_{\beta=1,2}$ [19, 20].

We now turn to a microscopic crossover region of $\alpha \to 0$ and Im $z \to 0$, where a precursor of the real eigenvalues of $\mathrm{GGE}_{\beta=1}$ is visible. We set a microscopic scaling near the real axis z $= i\eta N^{-1/4}$ and the crossover parameter near zero $\alpha = aN^{-1/4}$, which yield an asymptotic formula:

$$\overline{D_\tau} \sim e^{-a^4/2} e^{-\frac{2a^2\eta^2}{\tau}} \operatorname{erfc}\left(\frac{\sqrt{2}\,\eta^2}{\tau} - \frac{a^2}{\sqrt{2}}\right).$$

It shows an error function type bump near $\eta = 0$, which we interpret as the discussed precursor of an emerging bulk of real eigenvalues.

4. Conclusions

The method presented here is applicable to the Gaussian random matrix models for all $\beta = 1, 2, 4$ and serves as a tool for obtaining averages of both ratios and products of characteristic polynomials. Its main goal is to find a dual diffusion equation in the parameter space when the matrix itself undergoes a similar diffusive motion.

We calculated several examples for $\beta = 2$ GUE and GGE, where the resulting dual diffusion equations were particularly simple. We found a novel duality formula for products of characteristic polynomials, for $\mathrm{GGE}_{\beta=2}$, a previously not considered characteristic polynomial for the product of two $\mathrm{GGE}_{\beta=2}$ matrices and a $\beta = 1/\beta = 2$ Girko-Ginibre ensemble crossover model. We also dealt with $\mathrm{GGE}_{\beta=2}$ with variance structure and re-derived the ratio of characteristic polynomials in GUE case.

The main advantage of this method is a large reduction in the degrees of freedom. It also has a built-in external source random matrix models, which is especially suitable when looking for duality formulas of type (40).

Acknowledgments

Author acknowledges the support of the Grant DEC-2011/02/A/ST1/00119 of the National Centre of Science and the Australian Government Endeavour Fellowship during which this work was done. He would also thank P J Forrester, M A Nowak and P Warchol for reading the manuscript and valuable comments.

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